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THE MATHEMATICAL DEVELOPMENT OF THE END-POINT METHOD

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By S. Frankel and S. Goldberg

ABSTRACT

The end-point method is mathematically developed and its application to the Milne kernel studied in detail. The general solution of the Wiener-Hopf integral equation is first obtained. The Milne kernel appears in applying this method to the integral equation describing the diffusion and multiplication of neutrons in multiplying and scattering media. The neutrons are treated as monochromatic, isotropically scattered and of the same total mean free path in all materials involved. Only problems with spherical symmetry are treated, these being reducible to equivalent infinite slab problems. Solutions are obtained for tamped and untamped spheres; in the former case both growing and decaying exponential asymptotic solutions in the tamper are treated in detail. Appendix I treats the effects of the approximations inherent in the end-point method (cf. LADC - 79). Appendix II gives the solution of the inhomogeneous Wiener-Hopf equation.

INTRODUCTION

The general development of the end-point method and some of its applications are described in LADC - 79. It is the purpose of this report to supplement this general description with an explicit mathematical development of the end-point method and a detailed study of its application to the Milne kernel. This is the kernel entering in the integral equation describing the diffusion and multiplication of neutrons in multiplying and scattering materials where the neutrons are treated as monochromatic, isotropically scattered, and of the same total mean free path in all materials involved. The end-point method of treatment of integral equations is restricted to one-dimensional cases. This essentially limits the method to the treatment of problems in which the materials involved and the neutron distribution are both spherically symmetric, these problems being reducible to equivalent infinite-slab problems. In LADC - 79 it was shown that the end-point results may be applied loosely to problems of somewhat more complicated geometry and give more or less accurate approximations to the truth. These applications depend primarily on loose analogies rather than mathematical argument and will not be treated here.

Much of this report will be, in part, repetition of material treated in LADC 79. Here the emphasis will be primarily on the clear mathematical development of the methods of application presented there.

Chapter I

THE WIENER-HOPF METHOD

The integral equation,

$$n(x) = \int_0^\infty dx' \, n(x') \, K(x - x') \tag{1.0}$$

is known as the equation of Wiener and Hopf. With certain reasonable restrictions on the character of K and n this equation can be solved exactly. Before examining the method of solving this equation developed by Wiener and Hopf, it is useful to examine the simpler equation,

$$n(x) = \int_{-\infty}^{\infty} dx' \; n(x') \, K(x - x') \tag{1.1}$$

Since this equation is homogeneous, if $n_0(x)$ is a solution then $an_0(x)$ also satisfies the equation for any constant, a. Because of the infinite limits of integration and the "displacement" character of the kernel (K depends only on the difference, $x - x'$), $n_0(x - b)$ must also be a solution. If the solution, $n_0(x)$, is unique (except for a multiplicative factor) then $n_0(x - b) = an_0(x)$ for some a. Hence $n_0(x) = e^{kx}$. This suggests looking for exponential solutions of (1.1).

$$\begin{aligned} n(x) = e^{kx} &= \int_{-\infty}^{\infty} dx' \; e^{kx'} \; K(x - x') \\ &= e^{kx} \int_{-\infty}^{\infty} dy \; e^{-ky} \; K(y) \end{aligned} \tag{1.2}$$

$$\int_{-\infty}^{\infty} dy \; e^{-y} \; K(y) = 1$$

Any solution of this "characteristic equation" gives a value of k for which e^{kx} satisfies equation 1.1. If there is more than one solution to the characteristic equation, then any linear combination of the exponentials determined by them will satisfy equation 1.1.

These considerations will be relevant to the study of the equation 1.0 if K decays rapidly for large $|y|$. If this is the case, for large x, equation 1.0 approximates equation 1.1, and it may be expected that with increasing x the solutions of equation 1.0 will approach asymptotically the exponential solutions of equation 1.1. If this is the case, the asymptotic exponential part of the solution of equation 1.0 may be separated from the remainder of the solution by Laplace or Fourier transformation. The use of the Laplace transform is further suggested by the fact that the left hand term of equation 1.2 is the Laplace transform of the kernel.

Taking the Laplace transform of equation 1.1 gives:

$$\begin{aligned} \int_{-\infty}^{\infty} dx \; e^{-kx} \; n(x) &= \int_{-\infty}^{\infty} dx \; e^{-kx} \int_{-\infty}^{\infty} dx' \, n(x') \; K(x - x') \\ &= \int_{-\infty}^{\infty} dx' \; n(x') \, e^{-kx'} \int_{-\infty}^{\infty} dy \; e^{-ky} \; K(y) \end{aligned}$$

$$\int_{-\infty}^{\infty} dx \; e^{-kx} \; n(x) \left(\int_{-\infty}^{\infty} dy \; e^{-ky} \; K(y) - 1 \right) = 0$$

This last equation shows that the Laplace transform of n(x) must vanish for all values of k which do not satisfy the characteristic equation 1.2.

An application of the same technique to equation 1.0 does not lead immediately to a factored equation because of the finite lower limit. To get around this difficulty Wiener and Hopf introduced the following trick.

Define $n(x) \equiv f(x) + g(x)$

where $\quad f(x) \equiv 0$ for $x < 0$

$\qquad g(x) \equiv 0$ for $x \geqq 0$

This permits writing equation 1.0 in the form

$$f(x) + g(x) = \int_{-\infty}^{\infty} dx' \; f(x') \; K(x - x')$$

Now, taking the Laplace transform gives

$$\int_{-\infty}^{\infty} dx\, f(x)\, e^{-kx} + \int_{-\infty}^{\infty} dx\, g(x) e^{-kx} = \int_{-\infty}^{\infty} dx\, e^{-kx} \int_{-\infty}^{\infty} dx'\; f(x')\, K(x - x')$$

$$= \int_{-\infty}^{\infty} dx'\, e^{-kx'}\; f(x') \int_{-\infty}^{\infty} dy\, e^{-ky}\, K(y)$$

Defining

$$F(k) \equiv \int_{-\infty}^{\infty} dx\, f(x)\, e^{-kx}$$

$$G(K) \equiv \int_{-\infty}^{\infty} dx\, g(x)\, e^{-kx}$$

$$\underline{K}(k) \equiv \int_{-\infty}^{\infty} dx\, K(x) e^{-kx}$$

we have

$$G(k) = F(k)\left(\underline{K}(k) - 1\right) \equiv F(k)\; P(k) \tag{1.3}$$

This equation will hold for any value of k for which all three integrals exist. We therefore impose conditions on the kernel and solution of equation 1.0, which ensure the existence of a suitable region in the complex plane in which all three integrals exist. We require that K(y) decay at least as rapidly as an exponential for large (positive or negative) y.

$$K(y) = c(e^{-c|y|}),\; c > 0. \tag{1.4}$$

Then $\underline{K}(k)$ will exist for $-c < R(k) < c$. We further assume that

$$f(x) = c(e^{dx}),\; d < c \tag{1.5}$$

The kernels of primary interest are symmetric. For these, if the "largest" value of c satisfying equation 1.4 is chosen, equation 1.5 is not a restrictive condition, since f(x) must approach asymptotically an exponential, e^{kx}, for some k satisfying $\underline{K}(k) = 1$ and therefore within the range of convergence of $\underline{K}(k)$. The form of equation 1.3 clearly requires that g(x) decay (for large negative x) at least as fast as e^{cx}. Thus G(k) exists for all k having $R(k) < c$. The three integrals will therefore all exist throughout a vertical strip in the complex k-plane defined by $d < R(k) < c$.

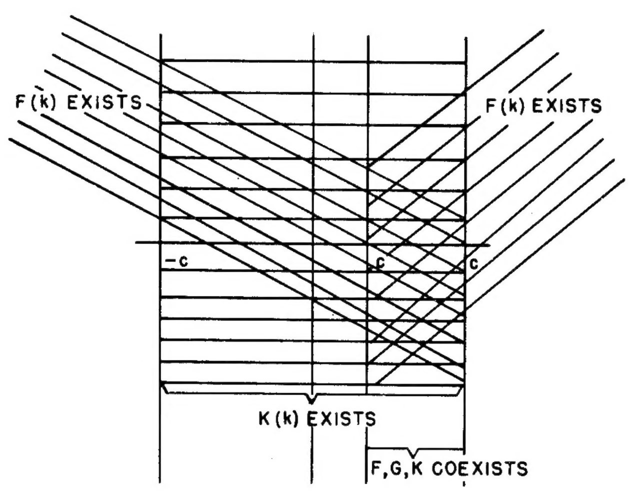


Figure 1.

Within this "common strip" all three integrals are convergent and equation 1.3 must be satisfied. Outside this strip the nonconvergent integrals will be defined by analytic extension (and need not be analytic) in such a way that the equation is still satisfied.

Within and to the right of the common strip, F(k) exists and is analytic. [It is clear from its definition that in this range any derivative of F(k) exists.] Similarly within and to the left of the strip, G(k) exists and is analytic. $\underline{K}$(k), hence also P(k), exists and is analytic within the strip but may have singularities on either side of it. We make the further assumption that F(k) and G(k) have no roots in their respective regions of analyticity. (Cf. Paley and Wiener, Fourier Transforms, p. 51). We further require that there exist a sub-strip within the common strip within which P(k) has no roots. [This must be true if P(k) has only a finite number of zeros in the common strip. This will actually be the case, Cf. Titchmarsh, Fourier Integrals, p. 339.]

We have now a sub-strip within which log P(k) is analytic; within which, and to the right, log F(k) is analytic; within which, and to the left, log G(k) is analytic, and within which the three satisfy

$$\log P(k) = \log G(k) - \log F(k)$$

This equation will be satisfied throughout the plane by the analytic extensions.

It is now easy to find functions, F and G, satisfying this equation and the analyticity conditions. For values of k within the sub-strip we express log P(k) by means of a Cauchy integral:

$$\log P(k) = (1/2\pi i) \int_C \frac{dk'}{k' - k} \log P(k')$$

$$= (1/2\pi i) \int_R \frac{dk'}{k' - k} \log P(k')$$

$$+ (1/2\pi i) \int_L \frac{dk'}{k' - k} \log P(k')$$

where the contour of integration consists of two vertical lines in the sub-strip, one running up to the right of k, the other down to its left.

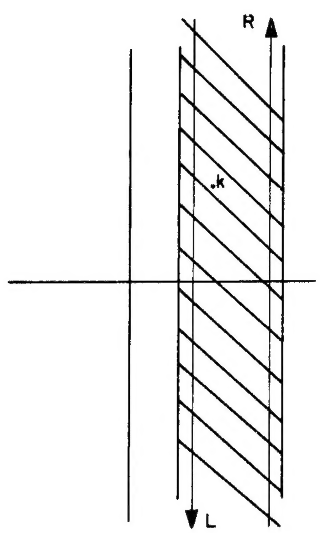


Figure 2.

We have now decomposed log P(k) into two parts, one certainly analytic within the strip and to the left, the other within and to the right. These may be identified with log G(k) and -log F(k), and give a solution to the equation 1.0.

$$\log F(k) = -\frac{1}{2\pi i}\int_L \frac{dk'}{k' - k}\log P(k) + \text{constant}$$

$$\log G(k) = \frac{1}{2\pi i}\int_R \frac{dk'}{k' - k}\log P(k') + \text{constant} \tag{1.6}$$

This contour integral representation of log F(k) determines F(k), hence also f(x).

$$f(x) = \frac{1}{2\pi i}\int_{\delta - i\infty}^{\delta + i\infty} e^{kx} F(k)\, dk \tag{1.7}$$

where δ is chosen to make F(k) regular along the contour. In particular, δ may be taken in the sub-strip. Since F(k) is analytic to the right of the sub-strip, the contour may be translated to the right as far as desired. For negative values of x this may be used to show that f(x) vanishes.

If f(x) contains a term $Ae^{k_o x}$ (e.g., as its asymptotic solution), then its Laplace transform, F(k) will contain a corresponding term.

$$\int_0^\infty dx\, e^{-kx} Ae^{k_o x} = A/(k - k_o)$$

Thus a pure exponential term in f(x) manifests itself in F(k) as a simple pole, and the coefficients of the two may be identified. The coefficient of the singularity is most easily determined by expanding log F(k) about the singularity.

$$\log F(k) = -\log(k - k_o) + \log A + 0(k - k_o)$$

The asymptotic solution will be determined by all of the singularities of F(k) on the imaginary axis and in the right half-plane. If there are no singularities on or to the right of the imaginary axis the solution, f(x), will approach zero asymptotically. A more useful asymptotic solution however, will be that determined by the first singularities to the left of the imaginary axis:

An important special case of this general treatment is that for which the kernel, K(y), is symmetric and for which the characteristic equation has only a single pair of conjugate roots on the imaginary axis. If these two roots are at $\pm\, ik_o$, then the solution will be of the form

$$F(k) = B\left[\sin k_o (x + x_o) + h(x)\right],\ h(x) \to 0 \text{ as } x \to +\infty \tag{1.8}$$

Since the equation is homogeneous, B is undetermined; x_o, however, can be evaluated.

$$F(k) = \int_0^\infty dx\, e^{-kx} B\left[\sin k_o(x + x_o) + h(x)\right]$$

$$= \int_0^\infty dx\, e^{-kx}\frac{B}{2i}\left[e^{ik_o(x + x_o)} - e^{-ik_o(x + x_o)} + 2ih(x)\right]$$

$$= \frac{B}{2i}\left(\frac{e^{ik_o x_o}}{k - ik_o} - \frac{e^{-ik_o x_o}}{k + ik_o} + 2iH(k)\right)$$

In the neighborhood of $\pm\, ik_o$, H(k) is finite. We expand log F(k) near these two poles,

$$\log F(ik_o + \epsilon) = \log\frac{B}{2i} + i\,k_o x_o - \log\epsilon + 0(\epsilon)$$

$$\log F(-\,ik_o + \epsilon) = \log\frac{-B}{2i} - i\,k_o x_o - \log\epsilon + 0(\epsilon)$$

$$\lim_{\epsilon \to 0}\left[\log F(ik_o + \epsilon) - \log F(-\,ik_o + \epsilon)\right] = \log(-1) + 2i\,k_o x_o$$

$$\log F(k) = \log G(k) - \log P(k)$$

$$= \frac{1}{2\pi i}\int_R \frac{dk'}{k' - k} \log P(k') - \log P(k)$$

$$\lim_{\epsilon \to 0}\left[\log P(ik_o + \epsilon) - \log P(-ik_o + \epsilon)\right] = \log\left[\frac{P'(ik_o)}{P'(-ik_o)}\right] = \log(-1)$$

since K(y) is even, K(k) and P(k) are even; P′(k) is odd.

$$2\, ik_o x_o = \frac{1}{2\pi i}\int_R dk' \log P(k')\left[\frac{1}{k' - ik_o} - \frac{1}{k' + ik_o}\right]$$

$$x_o = \frac{1}{2\pi i}\int_R \frac{dk'}{k'^2 + k_o^2} \log P(k') \qquad (1.9)$$

The two terms, log (- 1), have been neglected since the form of the solution 1.8 is unchanged by the addition of a multiple of π to $k_o x_o$. The evaluation of x_o completes the determination of the asymptotic form of the solution equation 1.8. x_o is expressed in equation 1.9 as a single integral, which in many cases must be evaluated numerically. To get the complete solution requires two integrations, one to evaluate log F(k) by equation 1.6, another to get f(x) by (1.7).

Two-Medium Problems

A more general problem that can be treated by the Wiener-Hopf technique is

$$n(x) = \int_{-\infty}^{0} dx'\ K'(x - x')\, n(x') + \int_{0}^{\infty} dx'\ K(x - x')\, n(x').$$

Breaking up n(x) as before and taking the Laplace transform of the resulting equation gives

$$F(k) + G(k) = \underline{K}(k)\, F(k) + \underline{K}'(k)\, G(k)$$

where the notation is the same as before. This may be written as

$$G(k) = F(k)\left(\frac{1 - \underline{K}(k)}{\underline{K}'(k) - 1}\right) \equiv F(k)\, P(k)$$

This is now of the same form as equation 1.3. The rest of the treatment proceeds in the same way. With this more complicated form for P(k) there may be a greater number of singularities of log P(k), leading to a larger number of independent solutions. In particular it is no longer necessary to require that g(x) decay exponentially away from the boundary.

An important special case of this two-medium problem is that for which K(y) and K′(y) differ only by a multiplicative factor. This case will be treated extensively in the second chapter.

The Wiener-Hopf technique may be further extended to permit the solution of inhomogeneous displacement integral equations. This method is outlined in Appendix II.

Chapter II

APPLICATION TO NEUTRON PROBLEMS

In this chapter we treat the applications of the Wiener-Hopf method (combined with some approximations) to problems concerning the spatial distribution and time dependence of neutrons in spheres of multiplying and scattering materials. It will be shown that such problems, with suitable physical approximations, can be represented by integral equations closely analogous to the Wiener-Hopf equation. By making suitable mathematical approximations (the "end-point method") fairly accurate solutions to these equations can be gotten from the corresponding Wiener-Hopf solutions.

We make the following physical approximations:

A) We consider only one neutron velocity; hence for each material only one value for each cross section.

B) We treat all collision processes as isotropic. (Anisotropy of elastic scattering can be treated to a limited extent. It can be shown that if this anisotropy is neglected and the transport average used for the elastic scattering cross section quite accurate results will be obtained. Cf. LADC - 79 and MT - 26.)

C) The total mean free path will be taken to be the same for all materials involved.

D) The neutron distribution will be treated as a continuum. It will be taken to be spherically symmetric and of stable spatial distribution. These three conditions will certainly be good approximations if the neutron distribution has lived through many generations and consists of a sufficient number of neutrons to make statistical fluctuation negligible.

We adopt the following notation:

σ_f is the fission probability per unit path length. (It is therefore the product of the fission cross section and the number of nuclei per unit volume.) Similarly,

σ_s is the scattering probability per unit path length.

σ_a is the absorption probability per unit path length.

$\sigma = \sigma_f + \sigma_s + \sigma_a$

ν is the mean number of neutrons emerging from a fission process.

$F = 1 + f = \dfrac{\nu\sigma_f + \sigma_s}{\sigma}$ is therefore the mean number of neutrons emerging from a collision.

v is the neutron velocity.

$n(\underline{r}, t)$ is the neutron density at point $\underline{r}$ at time t.

We express the neutron density at $(\underline{r}, t)$ as an integral over all points at which these neutrons may have suffered their last collisions.

$$v\, n(\underline{r}, t) = \int d\underline{r}'\ \sigma v\ F(\underline{r}')\ n\left(\underline{r}', t - \frac{|r - r'|}{v}\right) \frac{1}{4\pi(r - r')^2}\, e^{-\sigma|r - r'|} \tag{2.1}$$

We look for solutions of the form

$$n(\underline{r}, t) = n(\underline{r})\, e^{\gamma_0 t}$$

The integral equation 2.1, then takes the form:

$$n(\underline{r}) = \int d\underline{r}'\sigma\ F(\underline{r}')\ n(\underline{r}')\ \frac{1}{4\pi(r - r')^2}\ e^{-(\sigma + \gamma_0/v)\,|r - r'|}$$

We now rescale $\underline{r}$, taking as the unit of length the mean attenuation distance, $1/(\sigma + \gamma_0/v)$.

$$\underline{x} = \underline{r}\ (\sigma + \gamma_0/v)$$

$$n(\underline{x}) = \frac{1}{1 + \gamma_0/\sigma v} \int d\underline{x}'\ F(\underline{x}')\ n(\underline{x}')\ \frac{e^{-|x - x'|}}{4\pi(x - x')^2}$$

Defining $\gamma = \gamma_0/\sigma v$ gives the three-dimensional integral equation

$$n(\underline{x}) = \frac{1}{1 + \gamma} \int d\underline{x}'\ F(\underline{x}')n(\underline{x}')\ \frac{e^{-|x - x'|}}{4\pi(x - x')^2} \tag{2.2}$$

If we now introduce polar coordinates, $\underline{x}' = (r', \phi', \theta')$,

taking the point $\underline{x}$ on the polar axis we may make use of the assumed spherical symmetry of $n(\underline{x}')$ to reduce equation 2.2 to an equation in one dimension.

$$n(r) = \frac{1}{1 + \gamma} \int r'^2\, dr'\ F(r')\ n(r') \iint d\phi \sin\theta\, d\theta\, \frac{e^{-(r^2 + r'^2 - 2rr'\cos\theta)^{1/2}}}{4\pi(r^2 + r'^2 - 2rr'\cos\theta)}$$

Taking $\mu = \cos\theta$, $l^2 = r^2 + r'^2 - 2rr' \cos\theta$

$$\int_0^{2\pi} d\phi \int_0^{\pi} \sin\theta \, d\theta \, \frac{e^{-(r^2 + r'^2 - 2rr' \cos\theta)^{\frac{1}{2}}}}{4\pi(r^2 + r'^2 - 2rr' \cos\theta)} = \frac{1}{2} \int_{-1}^{1} d\mu \frac{e^{-l}}{l^2}$$

$$= \frac{1}{2} \int_{|r - r'|}^{r + r'} \frac{l \, dl}{rr'} \frac{e^{-l}}{l^2}, \left(d\mu = - \frac{l \, dl}{rr'}\right)$$

$$= \frac{1}{2rr'} \left[E(|r - r'|) - E(r + r')\right]$$

where $E(s) = \int_s^{\infty} \frac{e^{-t} dt}{t}$

$$r \, n(r) = \frac{1}{2(1 + \gamma)} \int_0^{\infty} dr' \, F(r') \, r' \, n(r') \left[E(|r - r'|) - E(r + r')\right] \tag{2.3}$$

If we now define $u(r) \equiv r \, n(r)$ and treat $u(r)$ as an odd function, and $F(r)$ as an even function of r [no meaning has previously been assigned to negative values of r or to the corresponding $n(r)$ and $F(r)$] , we may write equation 2.3 in the form:

$$u(r) = \frac{1}{2(1 + \gamma)} \int_{-\infty}^{\infty} dr' \, F(r') \, u(r') \, E(|r - r'|) \tag{2.4}$$

If instead of assuming the material and neutron distribution spherically symmetric, we take both as functions of only one Cartesian coordinate, z, equation 2.2 may be reduced to an equation in one dimension as follows:

$$n(z) = \frac{1}{1 + \gamma} \int dz' \, F(z') \, n(z') \iint dx' \, dy' \, \frac{e^{-\left[(z - z')^2 + (y - y')^2 + (x - x')^2\right]^{\frac{1}{2}}}}{4\pi\left[(z - z')^2 + (y - y')^2 + (x - x')^2\right]}$$

$$= \frac{1}{1 + \gamma} \int dz' \, F(z') \, n(z') \int_0^{2\pi} d\phi \int_0^{\infty} \rho d\rho \, \frac{e^{-l}}{4\pi l^2}$$

where $l^2 = (z - z')^2 + \rho^2$, $l \, dl = \rho d\rho$

$$n(z) = \frac{1}{2(1 + \gamma)} \int dz' \, F(z') \, n(z') \, E(|z - z'|) \tag{2.5}$$

A comparison of equations 2.4 and 2.5 shows that the sphere problem 2.4 may be identified with a slab problem 2.5 in which the distribution of materials $F(z)$ across the slab is the same as that along a diameter of the sphere. Any odd solution of the slab problem, $n(z)$, may be identified with the quantity $u(r)$ in the sphere problem and conversely. The "fundamental mode" of the sphere for which $n(r)$ is everywhere positive corresponds to the "first harmonic" of the slab in which the neutron density takes on apparently meaningless negative values. For this reason, and because higher modes may be superimposed on the fundamental, we will treat the neutron density, $n(z)$, as a real quantity which may have either sign.

For a tamped sphere of core radius a, outer tamper radius b, and mean attenuation distances, the integral equation 2.4 takes the form

$$u(r) = \frac{1 + f_t}{1 + \gamma} \int_{-b}^{-a} dr' \, u(r') \frac{1}{2} E(|r - r'|)$$

$$+\frac{1+f_c}{1+\gamma}\int_{-a}^{a} dr'\ u(r')\frac{1}{2}E\left(|r-r'|\right)$$

$$+\frac{1+f_t}{1+\gamma}\int_{a}^{b} dr'\ u(r')\frac{1}{2}E\left(|r-r'|\right)$$

where f_c and f_t are the values of f in core and tamper respectively. This equation differs from the Wiener-Hopf equation in having four boundaries instead of one (or two for an untamped sphere). With more than one boundary no exact solution is known. We therefore resort to an approximation, namely to treat the behaviour of the solution near each boundary as if no other boundaries existed. It was shown in the first chapter that the solution of the one-boundary problem approaches, at large distances from the boundary, a solution of the problem with infinite limits. It is reasonable to expect that the solution of a two-boundary problem in which the boundaries are very far apart will behave in some middle region as a solution of the infinite-limits equation. If this is the case, we have only to combine two one-boundary solutions in such a way that their asymptotic components coincide. In a many-boundary problem, e.g., the tamped sphere, we apply this recipe in each region. This approximation method, the "end-point method", would seem, from the above argument, reasonably accurate only if the distances between boundaries are many mean attenuation distances. It is shown in Appendix I that the limit of reasonable accuracy is actually a few tenths of a mean attenuation distance. There is therefore good reason to believe that for sizes larger than that, the end-point method is sufficiently accurate.

In order to apply the end-point method we must first study the one-boundary problem with the "Milne kernel",

$$K(y) = c\frac{1}{2}\cdot E(|y|)$$

This kernel with c = 1 occurs in "the equation of E. A. Milne" describing the flow of radiation through the outermost layers of a star. We will, however, refer to it as the "Milne kernel" for all positive values of c. The general equation we have to study is then

$$n(x) = c'\int_{-\infty}^{0} dx'\ n(x')\frac{1}{2}E(|x-x'|) + c\int_{0}^{\infty} dx'\ n(x')\frac{1}{2}E\left(|x-x'|\right)$$

$$c = (1+f)/(1+\gamma).$$

Several cases arise. For a free surface, either the outer surface of a tamper or the surface of an untamped sphere, we take $c' = 0$. For an interface we take both c and c' positive. For the core material, c must be greater than 1 ($f>\gamma$); in the tamper, c - 1 may be of either sign.

We first treat the free-surface case.

$$n(x) = c\int_{0}^{\infty} dx'\ n(x')\frac{1}{2}E\left(|x-x'|\right)$$

The characteristic equation is

$$c\int_{-\infty}^{\infty} dy\,\frac{1}{2}E\left(|y|\right)e^{-ky} = (c/2)\int_{0}^{\infty} dy\left(e^{-ky}+e^{ky}\right)\int_{1}^{\infty}\frac{ds}{s}e^{-ys}$$

$$= (c/2)\int_{1}^{\infty}\frac{ds}{s}\left(\frac{1}{s+k}+\frac{1}{s-k}\right)$$

$$= c\int_{1}^{\infty}\frac{ds}{s^2-k^2}$$

$$= \frac{c}{2k} \log\left(\frac{1+k}{1-k}\right) = \frac{c}{k} \tanh^{-1} k = 1$$

If $c < 1$ we have two real roots, $\pm k_o$ such that $c = k/\tanh {}^{1}k_o$. If $c > 1$ we have two imaginary roots, $\pm i\, k_o$, such that $c = k_o/\tan^{-1}k_o$. In either case it can be shown that the characteristic equation has only two roots. In the latter case the asymptotic solution is a sinusoidal function of $k_o x$, in the former, a hyperbolic function. We will represent the phase of the asymptotic solution by the "extrapolated end-point," x_o, such that the asymptotic solution is the sine or hyperbolic sine of $k_o(x + x_o)$. We now follow through, explicitly, the method of solution outlined in Chapter 1.

$$n(x) \equiv f(x) + g(x) = c \int_{-\infty}^{\infty} dx'\, f(x')\, \frac{1}{2}\, E\,(|x - x'|)$$

$$f(x) = o \text{ for } x < o$$

$$g(x) \equiv o \text{ for } x \geqq o$$

$$F(k) + G(k) = \int_{-\infty}^{\infty} dx\, n(x)\, e^{-kx} = \int_{-\infty}^{\infty} dx\, e^{-kx} \int dx'\, f(x')\, \frac{c}{2}\, E(|x - x'|)$$

$$= \int_{-\infty}^{\infty} dx'\, f(x')\, e^{-kx'} \int_{-\infty}^{\infty} d\,y\, e^{-ky} \frac{c}{2}\, E(|y|)$$

$$= F(k)\, \frac{c}{2k} \log\left(\frac{1+k}{1-k}\right)$$

$$G(k) = F(k)\left\{\frac{c}{2k} \log\left(\frac{1+k}{1-k}\right) - 1\right\} \equiv F(k)\, P(k)$$

$P(k)$ has singularities only at ± 1. These singularities are branch points so that to make the function explicit we introduce cuts lying along the real axis from $-\infty$ to -1 and from $+1$ to $+\infty$. We treat first the case $c > 1$. The two roots of $P(k)$ are then pure imaginary, $\pm ik_o$. The singularities of $\log P(k)$ are ± 1 and $\pm ik_o$. We look for a $\log F(k)$, analytic to the right of the imaginary axis [corresponding to the sinusoidal asymptotic solution, $f(x)$], and a $\log G(k)$, analytic to the left of $+1$ [corresponding to a $g(x)$ decaying somewhat faster than e^{-x}], and satisfying

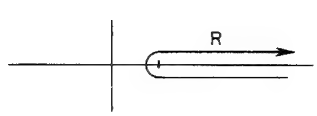
$$\log P(k) = \log G(k) - \log F(k) \qquad (2.6)$$

The "sub-strip" in which all three of these quantities are analytic is $0 < R(k) < 1$. We therefore break up $\log P(k)$ by means of a Cauchy integral along a contour running up and down in this strip and enclosing k, and (except for a common constant) identify $\log G(k)$ and $-\log F(k)$ with the two parts of the integral.

$$\log P_R(k) = \frac{1}{2\pi i} \int_R \frac{dk'}{k' - k} \log P(k') = \log G(k) + \text{constant},$$

$$\log P_L(k) = -\frac{1}{2\pi i} \int_L \frac{dk'}{k' - k} \log P(k') = \log F(k) + \text{constant}.$$

We simplify $\log P_R(k)$ by deforming the right contour to enclose the right-hand cut.



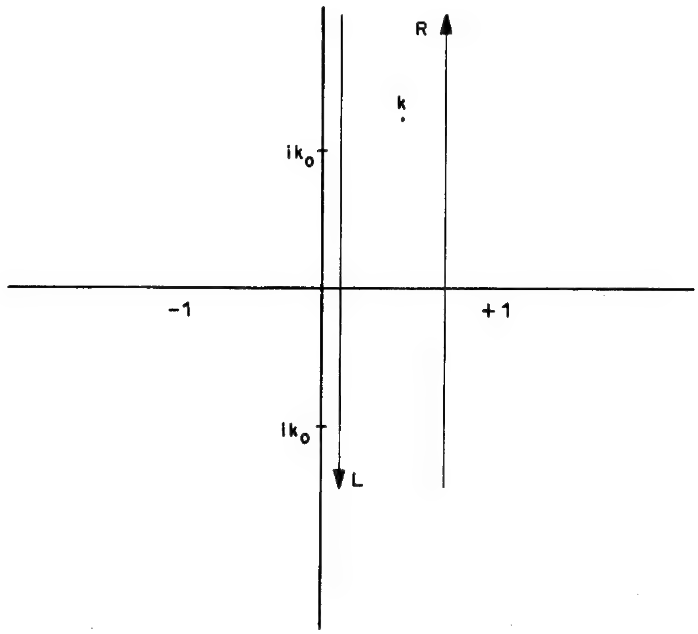


Figure 3.

$$\log P_R(k) = \frac{1}{2\pi i}\int_{\infty}^{\prime} \frac{dk'}{k' - k} \log\left[\frac{c}{2k'}\left(\log\frac{1+k'}{1-k'}\right) - 1\right] \quad \left[I(\log) \; = \pi i \rightarrow 0\right]$$

$$+ \frac{1}{2\pi i}\int_{1}^{\infty} \frac{dk'}{k' - k} \log\left[\frac{c}{2k'}\left(\log\frac{1+k'}{k'-1} + \pi i\right) - 1\right] \left[I(\log) = 0 \rightarrow + \pi i\right]$$

$$= \frac{1}{\pi}\int_{1}^{\infty} \frac{dk'}{k' - k} \tan^{-1}\left(\frac{\pi/2}{\frac{1}{2}\log\frac{k'+1}{k'-1} - \frac{k'}{c}}\right) \left[\tan^{-1} = 0 \rightarrow \pi\right]$$

Here the $\tan^{-1}$ rises from 0 at $k' = 1$ to π at $k' = +\infty$ (as indicated by the bracketed expressions). Substituting $k' = 1/s$,

$$\log P_R(k) = \frac{1}{\pi}\int_0^1 \frac{ds}{s(1 - ks)} T_c{}'$$

where

$$T_c = \tan^{-1}\left(\frac{\pi/2}{\tanh^{-1}s - 1/cs}\right) \quad \begin{array}{ll} T_c = \pi & s = 0 \\ = 0 & s = 1 \end{array}$$

$$\log P_R(k) = \frac{1}{\pi}\int_0^1 \frac{ds}{s} T_c + \frac{k}{\pi}\int_0^1 \frac{ds}{1 - ks} T_c \tag{2.7}$$

Here and throughout this treatment we encounter logarithmically infinite constants. A slight modification of our procedure [to make $P(k) \rightarrow 1$ as $|k| \rightarrow \infty$] suffices to avoid this embarrassment. The present treatment is somewhat simpler, though formally less rigorous.

We simplify log $P_L(k)$ by a corresponding deformation of the left contour.

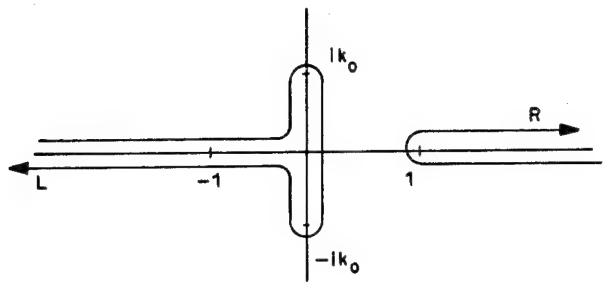


Figure 4.

$$- \log P_L(k) = \frac{1}{2\pi i} \left\{ \int_{-\infty}^{-1} \log \left[\frac{c}{2k'} \left(\log \frac{|k'| - 1}{1 - k'} + \pi i \right) - 1 \right] \quad \left[I(\log) = \pi i \rightarrow 2\pi i \right] \right.$$

$$+ \int_{-1}^{0} (2\pi i) + \int_{0}^{ik_o} (2\pi i) + \int_{-ik_o}^{0} (-2\pi i) + \int_{0}^{-1} (-2\pi i)$$

$$\left. + \int_{-1}^{-\infty} \log \left[\frac{c}{2k'} \left(\log \frac{|k'| - 1}{1 - k'} + \pi i \right) - 1 \right] \right\} \frac{dk'}{k' - k} \quad \left[I(\log) = -2\pi i \rightarrow -\pi i \right]$$

$$= \frac{1}{2\pi i} \int_{-\infty}^{-1} \frac{dk'}{k' - k} \log \frac{\frac{c}{2k'} \left(\log \frac{|k'| - 1}{1 - k'} + \pi i \right) - 1}{\frac{c}{2k} \left(\log \frac{|k'| - 1}{1 - k'} - \pi i \right) - 1} \quad \left[\log = 2\pi \rightarrow 4\pi \right]$$

$$+ \log \frac{k}{1 + k} + \log \frac{k - ik_o}{k} - \log \frac{k}{k + ik_o} - \log \frac{1 + k}{k}$$

Letting $r = -k'$

$$- \log P_L(k) = -\frac{1}{\pi} \int_{1}^{\infty} \frac{dr}{r + k} \tan^{-1} \frac{\pi/2}{\frac{r}{c} \frac{1}{2} \log \frac{r + 1}{r - 1}} \quad \left[\tan^{-1} = 2\pi \rightarrow \pi \right]$$

$$+ \log \frac{k^2 + k_o^2}{(2 + k)^2}$$

Letting $s = \frac{1}{r}$ we have

$$- \log P_L(k) = -\frac{1}{\pi}\int_0^1 \frac{ds}{s(1+ks)}\left[2\pi + \tan^{-1}\frac{\pi/2}{1/cs - \tanh^{-1}s}\right]\left[\tan^{-1} = -T_c = -\pi \to 0\right]$$

$$+ \log\frac{k^2 + k_o^2}{(1+k)^2}$$

$$= -2\int_0^1 \frac{ds}{s} + \frac{1}{\pi}\int_0^1 \frac{ds}{s(1+ks)} T_c + \log(k^2 + k_o^2)$$

$$\log P_L(k) = 2\int_0^1 \frac{ds}{s}\ \frac{1}{\pi}\int_0^1 \frac{ds}{s} T_c - \log(k^2 + k_o^2) + \frac{k}{\pi}\int_0^1 \frac{ds}{1+ks} T_c \tag{2.8}$$

Combining those two expressions, 2.7 and 2.8, with

$$\log P(k) \equiv \log\left(\frac{c}{2k}\log\frac{1+k}{1-k} - 1\right) = \log P_R(k) - \log P_L(k) \tag{2.6}$$

gives

$$\log\left(\frac{c}{2k}\ \log\frac{1+k}{1-k} - 1\right) = \frac{2}{\pi}\int_0^1 \frac{ds}{s}(T_c - \pi)$$

$$+ \log(k^2 - k_o^2) + \frac{2k^2}{\pi}\int_0^1 \frac{sds}{1 - k^2s^2} T_c \tag{2.9}$$

Taking the limit as $k \to 0$ we get

$$\frac{1}{\pi}\int_0^1 \frac{ds}{s}(T_c - \pi) = \frac{1}{2}\log\frac{c-1}{k_o^2} \tag{2.10}$$

and equation 2.9 becomes

$$\frac{k1}{\pi}\int_0^1 \frac{sds}{1 - k^2s^2} = \frac{k}{\pi}\int_0^1 \frac{ds}{1-ks} T_c - \frac{k}{\pi}\int_0^1 \frac{ds}{1+ks} T_c$$

$$= -\frac{1}{2}\log\left(\frac{k^2 + k_o^2}{k_o^2}\right) - \frac{1}{2}\log\frac{c-1}{\frac{c}{2k}\log\frac{1+k}{1-k} - 1} \tag{2.11}$$

Dividing by k^2 and again letting $k \to 0$,

$$\frac{1}{\pi}\int_0^1 sds\, T_c = -\frac{1}{2k_o^2} + \frac{c}{6(c-1)}$$

We now subtract the (infinite) constant, $2\int_0^1 \frac{ds}{s} - \frac{1}{\pi}\int_0^1 \frac{ds}{s} T_c - \log B$, from $\log P_R(k)$ and $\log P_L(k)$ to give $\log G(k)$ and $\log F(k)$.

$$\log F(k) = -\log(k^2 + k_o^2) + \frac{k}{\pi}\int_0^1 \frac{ds}{1+ks} T_c + \log B;$$

$$\log G(k) = \frac{2}{\pi}\int_0^1 \frac{ds}{s}\,(T_c - \pi) + \frac{k}{\pi}\int_0^1 \frac{ds}{1 - ks}\,T_c + \log B,$$

$$= \frac{k}{\pi}\int_0^1 \frac{ds}{1 - ks}\,T_c + \log\frac{B(c-1)}{k_o^2}$$

We now determine x_o and the value of B required to give the asymptotic sine wave in f(x) unit amplitude.

$$f(x) = \sin k_o(x + x_o) + h(x) \qquad h(x) \to 0 \text{ as } x \to +\infty$$

$$F(k) = \frac{e^{ik_ox_o}}{2i(k - ik_o)} - \frac{e^{-ik_ox_o}}{2i(k + ik_o)} + H(k) = \frac{k \sin k_ox_o + k_o \cos k_ox_o}{k^2 + k_o^2} + H(k)$$

$$\log F(ik_o + \epsilon) = -\log(2i) + ik_ox_o - \log\epsilon + 0(\epsilon)$$

$$\log F(-ik_o + \epsilon) = -\log(-2i) - ik_ox_o - \log\epsilon + 0(\epsilon)$$

$$\lim_{\epsilon\to 0}\left[\log F(ik_o + \epsilon) - \log F(-ik_o + \epsilon)\right] = \log(-1) + 2ik_ox_o$$

$$= \lim_{\epsilon\to 0}\left[\frac{ik_o + \epsilon}{\pi}\int_0^1 \frac{ds\ T_c}{1 + (ik_o + \epsilon)s} - \log(2ik_o\epsilon + \epsilon^2)\right.$$

$$\left. - \frac{ik_o + \epsilon}{\pi}\int_0^1 \frac{ds\ T_c}{1 + (-ik_o + \epsilon)s} + \log(-2ik_o\epsilon + \epsilon^2)\right]$$

$$= \frac{ik_o}{\pi}\int_0^1 ds\ T_c\left(\frac{1}{1 + ik_os} + \frac{1}{1 - ik_os}\right) + \log(-1)$$

$$x_o = \frac{1}{\pi}\int_0^1 \frac{ds}{1 + k_o^2s^2}\,T_c$$

Now adding the two values of log F gives

$$\log F(ik_o + \epsilon) + \log F(ik_o + \epsilon) = -2\log(2\epsilon) + 0(\epsilon),$$

$$= 2\log(2k_o\epsilon) + \frac{ik_o}{\pi}\int_0^1 ds\ T_c\left(\frac{1}{1 + ik_os}\cdot\frac{1}{1 - ik_os}\right) + 2\log B + 0(\epsilon)$$

$$= -2\log(2k_o\epsilon) + \frac{2k_o^2}{\pi}\int_0^2 \frac{s\ ds}{1 + k_o^2s^2}\,T_c + 2\log B + 0(\epsilon).$$

$$\log B = \log k_o = \frac{k_o^2}{\pi}\int_0^1 \frac{s\ ds}{1 + k_o^2s^2}\,T_c$$

This integral may be evaluated by allowing k to approach ik_o in equation 2.11:

$$-\frac{k_o^2}{\pi}\int_0^1 \frac{s\,ds}{1+k_o^2s^2}\,T_c = \lim_{\epsilon\to o}\left[-\frac{1}{2}\log\left(\frac{2ik_o\epsilon}{k_o^2}\right)-\frac{1}{2}\log\frac{c-1}{-\frac{1}{ik_o}\left(1-\frac{c}{1+k_o^2}\right)\epsilon}\right.$$

$$= -\frac{1}{2}\log\frac{2(c-1)}{1-\frac{c}{1+k_o^2}}$$

$$\log B = \frac{1}{2}\,\frac{k_o^2\left(1-\frac{c}{1+k_o^2}\right)}{2(c-1)}$$

$$\log F(k) = \frac{k}{\pi}\int_0^1 \frac{ds}{1+ks}\,T_c - \log(k^2+k_o^2) + \frac{1}{2}\log\frac{k_o^2\left(1-\frac{e}{1+k_o^2}\right)}{2(c-1)}$$

$$F(k) = \frac{k_o}{k^2+k_o^2}\sqrt{\frac{1-c/(1+k_o^2)}{2(c-1)}}\;e^{\frac{k}{\pi}\int_0^1\frac{ds}{1+ks}T_c}.$$

$$H(k)\;\frac{1}{k^2+k_o^2}\left(k_o\sqrt{\frac{1-c/(i+k_o^2)}{2(c-1)}}\;e^{\frac{k}{\pi}\int_0^1\frac{ds}{2+ks}T_c} - k\sin k_o x_o - k_o\cos k_o x_o\right)$$

We can evaluate $H(o)$, the total area of $h(x)$, and $\frac{-H\ (o)}{H(o)}$, its "mean length",

$$H(o) = \frac{1}{k_o}\left(\sqrt{\frac{1-c/(1+k_o^2)}{2(c-1)}} - \cos k_o x_o\right)$$

$$\frac{-H'(o)}{H(o)} = \frac{1}{H(o)k_o^2}\left(\sin k_o x_o - k_o\sqrt{\frac{1-c/(1+k_o^2)}{2(c-1)}} - \frac{1}{\pi}\int_0^1 ds T_c\right)$$

Making use of the formula

$$n(o) = \lim_{k\to\infty} k\int_0^\infty dx\, n(x)\, e^{-kx} = \lim_{k\to\infty} k\,F(k).$$

we get

$$n(0 = \lim_{k\to\infty}\frac{kk_o}{k^2+k_o^2}\sqrt{\frac{1-c/(1+k_o^2)}{2(c-1)}}\;e^{\frac{k}{\pi}\int_0^1\frac{ds}{1+ks}(T_c-\pi)+\log(1+k)}$$

$$n(o) = k_o\sqrt{\frac{1-c/(1+k_o^2)}{2(c-1)}}\;e^{\frac{1}{\pi}\int_0^1\frac{ds}{s}(T_c-\pi)} = \sqrt{\frac{1-c/(1+k_o^2)}{2}}.$$

We can derive an expression for $h(x)$ suitable for numerical evaluation as follows:

$$h(x) = \frac{1}{2\ i}\int_{-i\infty+\delta}^{i\infty+\delta} dk\; e^{kx}\,H(k), \qquad 0<\ \delta\ <1$$

H(k) is not singular at $\pm ik_o$. The bracketed expression vanishes, thus the contour may be deformed to lie along the left cut. Only the integral

$$\frac{k}{\pi}\int_0^1 \frac{ds}{1+ks}\,T_c = \frac{k}{\pi}\int_0^1 \frac{ds}{1-ks}\,T_c - \frac{2k^2}{\pi}\int_0^1 \frac{s\,ds}{1-k^2s^2}\,T_c$$

$$= \frac{k}{\pi}\int_0^1 \frac{ds}{1-ks}\,T_c - \log\left(\frac{k_o^2}{k^2+k_o^2}\,\frac{\frac{c}{2k}\log\frac{1+k}{1-k}-1}{c-1}\right)$$

is double-valued across the cut. Thus only the first term in H(k) contributes.

$$h(x) = \frac{1}{2\pi i}\int_{-\infty}^{1} dk\, e^{kx}\,\frac{k_o}{k^2+k_o^2}\sqrt{\frac{1-c/(1+k_o^2)}{2(c-1)}}\; e^{\frac{k}{\pi}\int_0^1 \frac{ds}{1-ks}T_c}\;\frac{(c-1)(k^2+k_o^2)}{k_o^2}\left[\frac{1}{\frac{c}{2k}\left(\log\left|\frac{1+k}{1-k}\right|-\pi i\right)-1} - \frac{1}{\frac{c}{2k}\left(\log\left|\frac{1+k}{1-k}\right|+\pi i\right)-1}\right]$$

$$= \frac{c}{2k_o}\sqrt{\frac{c-1}{2}\left(1-\frac{c}{1+k_o^2}\right)}\int_{-\infty}^{-1} \frac{dk\, e^{kx+\frac{k}{\pi}\int_0^1 \frac{ds}{1-ks}T_c}}{k\left[\left(\frac{c}{2k}\log\left|\frac{1+k}{1-k}\right|-1\right)^2+\frac{\pi^2c^2}{4k^2}\right]}$$

Replacing k by -k gives

$$h(x) = -\frac{c}{2k_o}\sqrt{\frac{c-1}{2}\left(1-\frac{c}{1+k_o^2}\right)}\int_1^{\infty} \frac{k\,dk\, e^{-\frac{k}{\pi}\int_0^1 \frac{ds}{1+ks}T_c}}{\left(\frac{c}{2}\log\frac{k-1}{k+1}-k\right)^2+\left(\frac{\pi c}{2}\right)^2}\, e^{-kx}$$

(h(x) is negative for all x).

If $c < 1$ the roots of the characteristic equation are $\pm k_1$, where $c = k_1/\tanh^{-1}k_1$. The contours must now be taken as shown in Figure 5.

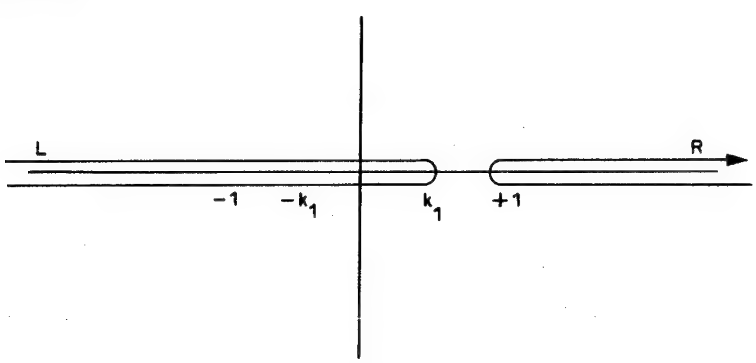


Figure 5.

Proceeding in the same way as for $c > 1$ we get the analogous results:

$$n(x) = \sinh k_1(x + x_0) + h(x)$$

$$\frac{1}{\pi}\int_0^1 \frac{ds}{s}\,(T_c - \pi) = \frac{1}{2}\log\frac{1-c}{k_1^2} \tag{2.12}$$

$$\frac{k^2}{\pi}\int_0^1 \frac{s\,ds}{1 - k^2 s^2}\,T_c = -\frac{1}{2}\log\frac{k_0^2 - k^2}{k_1^2}\cdot\frac{1-c}{1 - \frac{c}{2k}\log\frac{1+k}{1-k}} \tag{2.13}$$

$$x_0 = \frac{1}{\pi}\int_0^1 \frac{ds}{1 - k_1^2 s^2}\,T_c$$

$$T_c = \tan^{-1}\frac{\pi/2}{\tanh^{-1} s - 1/cs},\quad \left[\tan^{-1} = \pi \rightarrow 0\right]$$

$$F(k) = \frac{k_1}{k^2 - k_1^2}\sqrt{\frac{c/(1 - k_1^2) - 1}{2(1-c)}}\; e^{\frac{k}{F}\int_0^1 \frac{ds}{1+ks}T_c}$$

$$H(0) = -\frac{1}{k_1}\left[\sqrt{\frac{c/(1-k_1^2) - 1}{2(1-c)}} - \cosh k_1 x_0\right]$$

$$\frac{-H'(0)}{H(0)} = -\frac{1}{H(0)k_1^2}\left[\sinh k_1 x_0 - k_1\sqrt{\frac{c/(1-k_1^2) - 1}{2(1-c)}} - \frac{1}{\pi}\int_0^1 ds\,T_c\right]$$

$$n(0) = \sqrt{\frac{1}{2}\left(\frac{c}{1-k_1^2} - 1\right)}$$

$$h(x) = -\frac{c}{2k_1}\sqrt{\frac{1-c}{2}\left(\frac{c}{1-k_1^2} - 1\right)}\int_1^\infty \frac{k\,dk\, e^{-\frac{k}{\pi}\int_0^1 \frac{ds}{1+ks}T_c}}{\left(\frac{c}{2}\log\frac{k+1}{k-1} - 1\right)^2 + \left(\frac{\pi c}{2}\right)^2}\, e^{-kx}$$

Combining;these hyperbolic results ($c < 1$) with the elliptic results ($c > 1$) previously obtained shows the character of the solution and its numerically identifiable features to be continuous (as a function of c) across the parabolic ($c = 1$) boundary case.

We now treat the two-medium case, distinguishing the two materials (e.g., active material and tamper) only by their different values of c. Here four cases arise as the two c values are less than or greater than 1. We treat explicitly only the case: $c > 1$, $c' < 1$. The extension to other cases will then be obvious. Because of the applicability of the solution to the simple tamped sphere we refer to the one region, $c > 1$, $x > 0$, as "the core", and to the other, $c < 1$, $x < 0$, as "the tamper". We find two pertinent solutions, one belonging to a growing and the other to a decaying exponential asymptotic solution in the tamper. For the problem of the infinitely tamped sphere only the decaying solution will

figure (decaying as one moves away from the interface into the tamper). However, the "asymptotic solution" for a finite tamper will be a linear combination of the two solutions. The integral equation is:

$$n(x) = c' \int_{-\infty}^{0} dx'\, n(x') \frac{1}{2} E(|x - x'|) + c \int_{0}^{\infty} dx'\, n(x') \frac{1}{2} E(|x - x'|) \tag{2.14}$$

We use the same notation as before:

$$n(x) = f(x) + g(x)$$

$$f(x) = 0,\ x < 0$$

$$g(x) = 0,\ x \geqq 0$$

$$F(k) = \int_{-\infty}^{\infty} dx\, f(x) e^{-kx}$$

$$G(k) = \int_{-\infty}^{\infty} dx\, g(x) e^{-kx}$$

$$\underline{K}(k) = \int_{-\infty}^{\infty} dx \frac{1}{2} E(|x|)\, e^{-kx} = \frac{1}{2k} \log \frac{1 + k}{1 - k}$$

$$F(k) + G(k) = \int_{-\infty}^{\infty} dx\, n(x) e^{-kx}$$

$$= \int_{-\infty}^{\infty} dx\, e^{-kx} \int_{-\infty}^{\infty} dx' \frac{1}{2} E(|x - x'|) \left[c' g(x') + c f(x')\right]$$

$$= \int_{-\infty}^{\infty} dy\, e^{-ky} \frac{1}{2} E(|y|) \int_{-\infty}^{\infty} dx' e^{-kx'} \left[c' g(x') + c f(x')\right]$$

$$= \frac{1}{2k} \log \frac{1 + k}{1 - k} \left[c' G(k) + c F(k)\right]$$

$$G(k) = F(k) \frac{\frac{c}{2k} \log \frac{1 + k}{1 - k} - 1}{1 - \frac{c'}{2k} \log \frac{1 + k}{1 - k}} \equiv F(k) P(k)$$

The singularities of log P(k) now lie at:

± 1 (branch points)

$\pm ik_o \left[\text{roots of } P(k),\ \frac{k_o}{\tan^{-1} k_o} = c\right]$

$\pm k_1 \left[\text{poles of } P(k)\ \frac{k_1}{\tanh^{-1} k_1} = c'\right]$

$F(k)\left[\text{and we assume also } \log F(k)\right]$ must be analytic for $R(k) > 0$

$G(k)\left[\text{and we assume also } \log G(k)\right]$ must be analytic for

$R(K) < + k_1$ for "decaying solution", i.e., $g(x) = 0(e^{k_1 x})$

or $R(k) < - k_1$ for "growing solution", i.e., $g(x) = 0(e^{-k_1 x})$

log P(k) is analytic for $- 1 < R(k) < + 1$, except at $\pm ik_o$, $\pm k_1$

For the two cases we choose contours as follows:

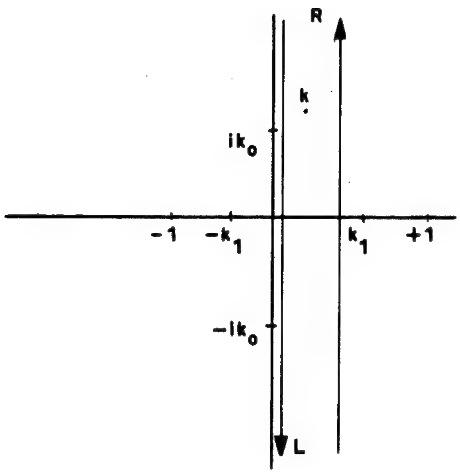


Figure 6.
"Decaying Solution"

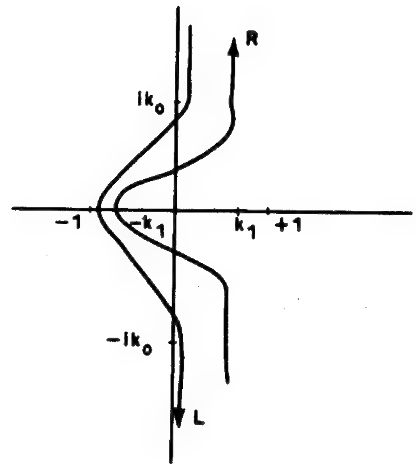


Figure 7.
"Growing Solution"

We treat first the decaying solution. As before we identify log F(k) and log G(k) with the left and right integrals (again excepting a constant).

$$\log P_R(k) = \frac{1}{2\pi i} \int_R \frac{dk'}{k' - k} \log P(k') = \log G(k) + \text{const.}$$

$$\log P_L(k) = -\frac{1}{2\pi i} \int_L \frac{dk}{k' - k} \log P(k') = \log F(k) + \text{const.}$$

We deform the contours as follows:

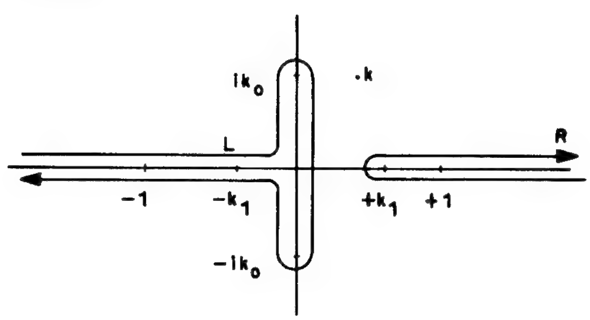


Figure 8.

$$\log P_R(k) = \frac{1}{2\pi i} \int_R \frac{dk'}{k' - k} \left[\log\left(\frac{c}{2k'} \log \frac{1 + k'}{1 - k'} - 1\right) - \log\left(1 - \frac{c'}{2k'} \log \frac{1 + k'}{1 - k'}\right)\right]$$

$$= \frac{1}{\pi} \int_0^1 \frac{ds}{s(1 - ks)} \, Tc - \frac{1}{2\pi i} \int_R \frac{dk'}{k' - k} \log\left(1 - \frac{c'}{2k'} \log \frac{1 + k'}{1 - k'}\right) \qquad (2.15)$$

making use of the previous evaluation of the first term.

$$\log P_R(k) = \frac{1}{\pi}\int_0^1 \frac{ds}{s(1-ks)}\, T_c - \frac{1}{2\pi i}\int_{k_1}^{\infty} \frac{dk'}{k'-k}(-2\pi i) - \frac{1}{2\pi i}\int_{R'} \frac{dk'}{k'-k}\log\left(\frac{c'}{2k'}\log\frac{1+k'}{1-k'} - 1\right)$$

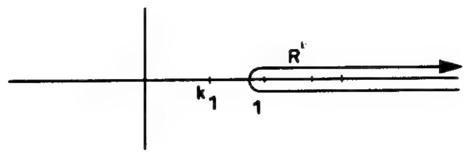


Figure 9.

The last integral is now equivalent to that evaluated in equation 2.15 (and is identical with the right-contour integral occurring in the one-medium problem for $c < 1$).

$$\log P_R(k) = \frac{1}{\pi}\int_0^1 \frac{ds}{s(1-ks)}\, T_c + \int_0^{1/k_1} \frac{ds}{s(1-ks)} - \frac{1}{\pi}\int_0^1 \frac{ds}{s(1-ks)}\, T_{c'}$$

$$= \frac{k}{\pi}\int_0^1 \frac{ds}{1-ks}(T_c - T_{c'}) + \frac{1}{\pi}\int_0^1 \frac{ds}{s}(T_c - T_{c'}) + \int_0^{1/k_1} ds\left(\frac{1}{s} + \frac{k}{1-ks}\right)$$

We choose the constant to make

$$\log G(k) = \log P_R(k) + \log B - \frac{1}{\pi}\int_0^1 \frac{ds}{s}(T_c - T_{c'}) - \int_0^{1/k_1} \frac{ds}{s}$$

$$= \frac{k}{\pi}\int_0^1 \frac{ds}{1-ks}(T_c - T_{c'}) + \log\frac{Bk_1}{k_1 - k} \tag{2.16}$$

Evaluating the left-contour integral gives

$$-\log P_L(k) = \frac{1}{2\pi i}\int_L \frac{dk'}{k'-k}\left[\log\left(\frac{c}{2k'}\log\frac{1+k'}{1-k'} - 1\right) - \log\left(1 - \frac{c'}{2k'}\log\frac{1+k'}{1-k'}\right)\right]$$

$$= \left\{-2\int_0^1 \frac{ds}{s} + \log(k^2 + k_0^2) + \frac{1}{\pi}\int_0^1 \frac{ds}{s(1+ks)}\, T_c\right\}$$

$$- \frac{1}{2\pi i}\int_{L'} \frac{dk'}{k'-k}\log\left(1 - \frac{c'}{2k'}\log\frac{1+k'}{1-k'}\right)$$

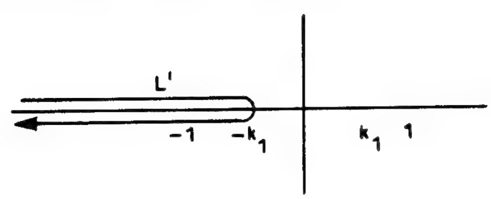


Figure 10.

$$= \{\ldots\} - \frac{1}{2\pi i}\int_{-\infty}^{-k_1} \frac{dk'}{k'-k}(2\pi i) - \frac{1}{2\pi i}\int_{L''} \frac{dk'}{k'-k}\log\left(\frac{c'}{2k'}\log\frac{1+k'}{1-k'} - 1\right)$$

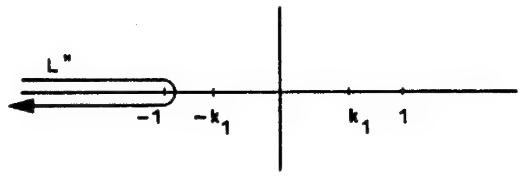


Figure 11.

$$\frac{1}{2\pi i}\int_{L''}\frac{dk}{k'-k}\log\left(\frac{c'}{2k'}\log\frac{1+k'}{1-k'}-1\right)=\frac{1}{2\pi i}\int_{R'}\frac{dk''}{k''+k}\log\left(\frac{c}{2k''}\log\frac{1+k''}{1-k''}-1\right),$$

$$=\frac{1}{\pi}\int_0^1\frac{ds}{s(1+ks)}\,T_{c'}\,.$$

$$-\log P_L(k)=-2\int_0^1\frac{ds}{s}+\log(k^2+k_o^2)+\frac{1}{\pi}\int_0^1\frac{ds}{s(1+ks)}\,T_c+\int_0^{1/k_2}ds\left(\frac{1}{s}-\frac{k}{1+ks}\right)-\frac{1}{\pi}\int_0^1\frac{ds}{s(1+ks)}\,T_{c'}$$

$$=-\frac{k}{\pi}\int_0^1\frac{ds}{1+ks}(T_c-T_{c'})+\log\frac{k_1(k^2+k_o^2)}{k_1+k}+\frac{1}{\pi}\int_0^1\frac{ds}{s}(T_c-T_{c'})-2\int_0^1\frac{ds}{s}+\int_0^{1/k_1}\frac{ds}{s}$$

$$\log F(k)=\log P_L(k)+\log B-\frac{1}{\pi}\int_0^1\frac{ds}{s}(T_c-T_c)-\int_0^{1/k_1}\frac{ds}{s}$$

$$=\frac{k}{\pi}\int_0^1\frac{ds}{1+ks}(T_c-T_{c'})+\log\frac{(k_1+k)B}{k_1(k^2+k_o^2)}-\frac{2}{\pi}\int_0^1\frac{ds}{s}(T_c-T_{c'})+2\int_{1/k_1}^1\frac{ds}{s}$$

$$-\frac{2}{\pi}\int_0^1\frac{ds}{s}\Big[(\pi-T_{c'})-(\pi-T_c)\Big]=-\log\frac{k_1^2}{1-c'}+\log\frac{k_o^2}{c-1}$$

$$\log F(k)=\frac{k}{\pi}\int_0^1\frac{ds}{1+ks}(T_c-T_{c'})+\log\frac{(k_1+k)B}{k_1(k^2+k_o^2)}+\log\left(\frac{1-c'}{k_1^2}\cdot\frac{k_o^2}{c-1}\right)+\log k_1^2$$

$$=\frac{k}{\pi}\int_0^1\frac{ds}{1+ks}(T_c-T_{c'})+\log\frac{Bk_o^2(k_1+k)(1-c')}{k_1(k^2+k_o^2)(c-1)}\,.$$

We again determine x_o and the value of B required to make the asymptotic sine solution of unit amplitude.

$$f(x) = \sin k_o(x + x_o) + h(x),\ x > 0,\ h(x) \to 0 \text{ as } x \to +\infty \tag{2.17}$$

$$F(k) = \frac{1}{2i}\left(\frac{e^{ik_ox_o}}{k - ik_o} - \frac{e^{-ik_ox_o}}{k + ik_o}\right) + H(k)$$

$$\lim_{\epsilon \to 0}\left[\log F(ik_o + \epsilon) - \log F - ik_o + \epsilon)\right] = \log\ (-1) + 2ik_ox_o$$

$$= \frac{2ik_o}{\pi}\int_0^1 \frac{ds}{1 + k_1{}^2s^2}\,(T_c - T_{c'}) + \log\left(\frac{-2ik_o\epsilon}{+2ik_o\epsilon}\right) + \log\frac{k_1 + ik_o}{k_1 - ik_o}$$

$$x_o = \frac{1}{\pi}\int_0^1 \frac{ds}{1 + k_o{}^2s^2}\,(T_c - T_{c'}) + \frac{1}{k_o}\tan^{-1}\frac{k_o}{k_1} = x_1 + \frac{1}{k_o}\tan^{-1}\frac{k_o}{k_1} \tag{2.18}$$

$$\lim_{\epsilon \to 0}\left[\log F(ik_o + \epsilon) + \log F(-ik_o + \epsilon) - 2\log\epsilon\right] = -2\log 2$$

$$= \frac{2k_o{}^2}{\pi}\int_0^1 \frac{s\,ds}{1 + k_o{}^2s^2}\,(T_c - T_{c'}) + 2\log\frac{Bk_o{}^2(1 - c')}{k_1(c - 1)} + \log\frac{k_1{}^2 + k_o{}^2}{4k_o{}^2}$$

The first term may be evaluated by the use of equation 2.11 and equation 2.13.

$$\frac{2k_o{}^2}{\pi}\int_0^1 \frac{sds}{1 + k_o{}^2s^2}\,(T_c - T_{c'}) = \lim_{\epsilon \to 0}\left[\log\left\{\frac{2ik_o\epsilon(c - 1)}{k_o{}^2\frac{i}{k_o}\left(1 - \frac{c}{1 + k_o{}^2}\right)\epsilon}\right\}\right] - \log\frac{(k_1{}^2 + k_o{}^2)(1 - c')}{k_1{}^2\left(1 - \frac{c'}{k_o}\tan^{-1}k_o\right)}$$

$$= \log\frac{2(c - 1)k_1{}^2(1 - c'/c)}{\left(1 - \frac{c}{1 + k_o{}^2}\right)(k_1{}^2 + k_o{}^2)(1 - c')}\ . \tag{2.19}$$

$$\log B = \log\frac{k_1(c - 2)}{k_o{}^2(1 - c')} - \frac{1}{2}\log\frac{k_1{}^2 + k_o{}^2}{k_o{}^2}\ \frac{1}{2}\log\frac{2(c - 1)k_1{}^2(1 - c'/c)}{\left[1 - c/(1 + k_o{}^2)\right](k_1{}^2 + k_o{}^2)(1 - c')}$$

$$= \frac{1}{2}\log\frac{(c - 1)\left[1 - c/(1 + k_o{}^2)\right]}{2k_o{}^2(1 - c')(1 - c'/c)}$$

$$\log F(h) = \frac{k}{\pi}\int_0^1 \frac{ds}{1 + ks}\,(T_c - T_{c'}) + \frac{1}{2}\log\frac{k_o{}^2(1 - c')\left[1 - c/(1 + k_o{}^2)\right]}{2k_1{}^2(c - 1)(1 - c'/c)} + \log\left(\frac{k + k_1}{k^2 + k_o{}^2}\right)$$

$$F(k) = \frac{k_o}{k_1}\ \frac{k + k_1}{k^2 + k_o^2}\sqrt{\frac{(1 - c')\left[1 - c/(1 + k_o^2)\right]}{2(c - 1)(1 - c'/c)}}\ e^{\frac{k}{\pi}\int_o^1 \frac{ds}{1 + ks}(T_c - T_{c'})}$$

$$H(k) = F(k) - \frac{k \sin k_o x_o + k_o \cos k_o x_o}{k^2 + k_o^2}$$

$$= \frac{1}{k^2 + k_o^2}\left[\frac{k_o}{k_1}(k + k_1)\sqrt{\frac{(1 - c')\left[1 - c/(1 + k_o^2)\right]}{2(c - 1)(1 - c'/c)}}\ e^{\frac{k}{\pi}\int_o^1 \frac{ds}{1 + ks}(T_c - T_{c'})} - k \sin k_o x_o - k_o \cos k_o x_o\right]$$

$$H(0) = \frac{1}{k_o}\left[\frac{(1 - c')\left[1 - c/(1 + k_o^2)\right]}{2(c - 1)(1 - c'/c)} - \cos k_o x_o\right]$$

$$H'(0) = \frac{(1 - c')\left[1 - c/(2 + k_o^2)\right]}{2(c - 1)\ (1 - c'/c)}\left(\frac{1}{k_o k_1} + \frac{1}{k_o \pi}\int_o^1 ds\ (T_c - T_{c'})\right) - \frac{1}{k_o^2}\sin k_o x_o$$

$$-\frac{H'(0)}{H(0)} = -\frac{1}{H(0)\ k_o}\left[\sqrt{\frac{(1 - c')\left[1 - c/(1 + k_o^2)\right]}{2(c - 1)(1 - c'/c)}}\left(\frac{1}{k_1} + \frac{1}{\pi}\int_o^1 ds\ (T_c - T_{c'})\right) - \frac{1}{k_o}\sin k_o x_o\right]$$

$$n(0) = \lim_{k \to \infty} kF(k) = \lim_{k \to \infty}\frac{k}{k^2 + k_o^2}\ \frac{k_o}{k_1}\ (k + k_1)\sqrt{\frac{(1 - c')\left[1 - c/(1 + k_o^2)\right]}{2(c - 1)(1 - c'/c)}}\ e^{\frac{k}{\pi}\int_o^1 \frac{ds}{1+ks}(T_c - T_{c''})}$$

$$e^{\frac{k}{\pi}\int_o^1 \frac{ds}{1 + ks}(T_c - \pi + \pi - T_{c'})} \to e^{\frac{1}{\pi}\int_c^1 \frac{ds}{s}(T_c - \pi) - \frac{1}{\pi}\int_o^1 \frac{ds}{s}(T_{c'} - \pi)} = \sqrt{\frac{(c - 1)\ k_1^2}{k_o^2(1 - c')}}$$

(using (2.10) and (2.12)).

$$n(0) = \sqrt{\frac{1 - c/(1 + k_o^2)}{2(1 - c'/c)}} \tag{2.20}$$

$$h(x) = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} dk\ H(k)\ e^{kx}$$

$$= \frac{1}{2\pi i}\int_{-L''}\frac{dk\ e^{kx}(k + k_1)}{k^2 + k_o^2}\ Ce^{\frac{k}{\pi}\int_o^1 \frac{ds}{1 + ks}(T_c - T_{c'})}$$

Figure 12.

where $C = \frac{k_0}{k_1}\sqrt{\frac{\left[1 - c/(1+k_0^2)\right](1-c')}{2(c-1)(1-c'/c)}}$

$$e^{\frac{k}{\pi}\int_0^1 \frac{ds}{1+ks}(T_c - T_{c'})} = e^{\frac{k}{\pi}\int_0^1 \frac{ds}{1-ks}(T_c - T_{c'})}\frac{(k^2+k_0^2)(c-1)}{k_0^2\left(\frac{c}{2k}\log\frac{1+k}{1-k}-1\right)}\cdot\frac{k_1^2\left(1-\frac{c'}{2k}\log\frac{1+k}{1-k}\right)}{(k_1^2-k^2)(1-c')}.$$

$$k(x) = \frac{1}{2\pi i}\int_{\infty}^{-1} dk\, e^{kx}\, C\,\frac{k_1^2(c-1)}{k_0^2(k_1-k)(1-c')}\, e^{\frac{k}{\pi}\int_0^1\frac{ds}{1-ks}(T_c - T_{c'})}\left\{\frac{1-\frac{c'}{2k}\left(\log\left|\frac{1+k}{1-k}\right|-\pi i\right)}{\frac{c}{2k}\left(\log\left|\frac{1+k}{1-k}\right|-\pi i\right)-1}\right.$$

$$\left.-\frac{1-\frac{c'}{2k}\left(\log\left|\frac{1+k}{1-k}\right|+\pi i\right)}{\frac{c}{2k}\left(\log\left|\frac{1+k}{1-k}\right|+\pi i\right)-1}\right\}.$$

Replacing k by -k gives

$$h(x) = \frac{1}{2\pi i}\int_1^{\infty} dk\, e^{-kx}\,\frac{k_1}{k_0(k_1+k)}\sqrt{\frac{[1-c/(1+k_0^2)]\,(c-1)}{2(1-c')\,(1-c'/c)}}\; e^{-\frac{k}{\pi}\int_0^1\frac{ds}{1+ks}(T_c - T_{c'})}\left\{\cdots\right\}$$

where

$$\left\{\cdots\right\} = -\frac{2\pi i}{2k}\,\frac{c\left(1-\frac{c'}{2k}\log\frac{k+1}{k-1}\right)+c'\left(\frac{c}{2k}\log\frac{k+1}{k-1}-1\right)}{\left(\frac{c}{2k}\log\frac{k+1}{k-1}-1\right)^2+\left(\frac{c\pi}{2k}\right)^2} = -\frac{\pi i}{k}\,\frac{c-c'}{\left(\frac{c}{2k}\log\frac{k+1}{k-1}-1\right)^2+\left(\frac{c\pi}{2k}\right)^2}$$

$$h(x) = -\frac{k_1^2}{2k_0}\sqrt{\frac{\left[1-c/(1+k_0^2)\right](c-1)(1-c'/c)}{2(1-c')}}\int_1^{\infty}\frac{k\,dk}{k+k_1}\,\frac{e^{-\frac{k}{\pi}\int_0^1\frac{ds}{1+ks}(T_c - T_{c'})}}{\left(\frac{c}{2}\log\frac{h+1}{k-1}-k\right)^2+\left(\frac{c\pi}{2}\right)^2}\,e^{-kx}$$

Now returning to G(k)

$$\log G(k) = \frac{k}{\pi}\int_0^1\frac{ds}{1-ks}(T_c - T_{c'}) + \log\frac{Bk_1}{k_1-k} \tag{2.16}$$

$$= \frac{k}{\pi}\int_0^1\frac{ds}{1-ks}(T_c - T_{c'}) + \log\frac{k_1}{k_1-k} + \frac{1}{2}\log\frac{(c-1)\left[1-c/(1+k_0^2)\right]}{2k_0^2(1-c')(1-c'/c)}$$

A check of this expression is afforded by evaluating

$$g(-c) = \lim_{k \to -\infty} - k\, G(k) = \sqrt{\frac{1 - c/(1 + k_o^2)}{2(1 - c'/c)}} = n(0), \qquad \text{(ef. equation 2.20).}$$

$$G(k) = \int_{-\infty}^{0} dx\, e^{-kx} g(x) = \int_{-\infty}^{0} dx\, e^{-kx} \left[A e^{k_1 x} + j(x) \right],$$

$$\text{where } j(x) = 0(e^{k_1 x}) \text{ as } x \to -\infty$$

$$G(k) = \frac{A}{k_1 - k} + J(k), \ J(k_1) \text{ is finite.}$$

$$\log G(k_1 + \epsilon) = \log \left(\frac{-A}{\epsilon} \right) + 0(\epsilon)$$

$$= \log \left(\frac{-k_1}{s} \right) + \frac{k_1}{\pi} \int_0^1 \frac{ds}{1 - k_1 s} (T_c - T_{c'}) + \frac{1}{2} \log \frac{(c-1)\left[1 - c/(1 + k_o^2)\right]}{2k_o^2 (1 - c')(1 - c'/c)}$$

$$A = \frac{k_1}{k_o} \sqrt{\frac{(c - 1)\left[1 - c/(1 + k_o^2)\right]}{2(1 - c')(1 - c'/c)}}\; e^{\frac{k_1}{\pi} \int_0^1 \frac{ds}{1 - k_1 s} (T_c - T_{c'})}.$$

$$\frac{k_1}{\pi} \int_0^1 \frac{ds}{1 - k_1 s} (T_c - T_{c'}) = \frac{k_1}{\pi} \int_0^1 \frac{ds}{1 - k_1^2 s^2} (T_c - T_{c'}) + \frac{k_1^2}{\pi} \frac{s\, ds}{1 - k_1^2 s^2} (T_c - T_{c'}).$$

The first term will be called $k_1 x_2$ by analogy with the x_1 introduced in equation 2.18, the second can be evaluated by the use of equation 2.11 and 2.13.

$$e^{\frac{k_1}{\pi} \int_0^1 \frac{ds}{1 - k_1 s} (T_c - T_{c'})} = e^{k_1 x_2} \sqrt{\frac{2k_o^2 (c/c' - 1)(1 - c')}{(k_1^2 + k_o^2)(c - 1)\left[c'/(1 - k_1^2) - 1\right]}} \qquad (2.21)$$

so that

$$A = \frac{k_1}{\sqrt{k_1^2 + k_o^2}} \frac{c\left[1 - c/(1 + k_o^2)\right]}{c'\left[c'/(1 - k_1^2) - 1\right]} \quad e^{k_1 x_2}$$

$$g(x) = \frac{k_1 \sqrt{c\left[1 - c/(1 + k_o^2)\right]}}{\sqrt{k_1^2 + k_o^2}\, \sqrt{c'\left[c'/(1 - k_1^2) - 1\right]}} e^{k_1 (x + x_1)} + j(x) \qquad (2.22)$$

$$J(k) = G(k) - \frac{A}{k_1 - k}$$

$$= \frac{k_1}{k_o (k_1 - k)} \sqrt{\frac{(c - 1)\left[1 - c/(1 + k_o^2)\right]}{2(1 - c')(1 - c'/c)}} \left\{ e^{\frac{k}{\pi} \int_0^1 \frac{ds}{1 - ks} (T_c - T_{c'})} - e^{\frac{k_1}{\pi} \int_0^1 \frac{ds}{1 - k_1 s} (T_c - T_{c'})} \right\}$$

$$j(x) = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} dk\, e^{kx}\, J(k),$$

$$= \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} dk\, \frac{e^{kx}\, k_1}{k_0(k_1 - k)}\sqrt{\frac{(c-1)\left[1 - c/(1+k_0^2)\right]}{2(1-c')(1-c'/c)}}\left\{ e^{\frac{k}{\pi}\int_0^1 \frac{ds}{1-ks}(T_c - T_{c'})} - e^{\frac{k_1}{\pi}\int_0^1 \frac{ds}{1-k_1 s}(T_c - T_{c'})}\right\}$$

$$= \frac{1}{2\pi i}\,\frac{k_1}{k_0}\sqrt{\frac{(c-1)\left[1 - c/(1+k_0^2)\right]}{2(1-c')(1-c'/c)}}\int_{R'} \frac{dk}{k_1 - k}\, e^{kx}\, e^{\frac{k}{\pi}\int_0^1 \frac{ds}{1+ks}(T_c - T_{c'})}$$

$$\left\{ \frac{k_0^2(k_1^2 - k^2)(1-c')\left(\frac{c}{2k}\log\frac{1+k}{1-k} - 1\right)}{(k^2+k_0^2)(c-1)\,k_1^2\left(1 - \frac{c'}{2k}\log\frac{1+k}{1-k}\right)}\right\}$$

$$= \frac{1}{2\pi i}\,\frac{k_0}{k_1}\sqrt{\frac{(1-c')\left[1 - c/(1+k_0^2)\right]}{2(c-1)(1-c'/c)}}\int_1^\infty \frac{dk(k+k_1)\, e^{kx}\, e^{\frac{k}{\pi}\int_0^1 \frac{ds}{1+ks}(T_c - T_{c'})}}{k^2+k_0^2}\left\{\frac{\frac{c}{2k}\left(\log\frac{k+1}{k-1} + \pi i\right) - 1}{1 - \frac{c'}{2k}\left(\log\frac{k+1}{k-1} + \pi i\right)}\right.$$

$$\left. - \frac{\frac{c}{2k}\left(\log\frac{k+1}{k-1} - \pi i\right) - 1}{1 - \frac{c'}{2k}\left(\log\frac{k+1}{k-1} + \pi i\right)}\right\}$$

$$j(x) = \frac{k_0}{2k_1}\sqrt{\frac{(1-c')\left[1 - c/(1+k_0^2)\right](1-c'/c)}{2(c-1)}}\int_1^\infty \frac{k\, dk\,(k+k_1)}{k^2+k_0^2}$$

$$\frac{e^{\frac{k}{\pi}\int_0^1 \frac{ds}{1+ks}(T_c - T_{c'})}}{\left(k - \frac{c'}{2}\log\frac{k+1}{k-1}\right)^2 + \left(\frac{c'\pi}{2}\right)^2}\, e^{kx} \qquad (2.23)$$

The second solution differs in having as an asymptotic solution in the tamper a growing exponential (growing for increasing negative x), $e^{-k_1 x}$. The core solution is again sinusoidal, differing only in phase from the first solution. Thus, the left contour must still lie to the right of the roots of P(k) at $\pm ik_0$. The tamper solution, g(x), is to grow as $e^{-k_1 x}$. Thus G(k) must have a pole at $-k_1$. (It may also have a pole at $+k_1$, the corresponding asymptotic g(x), $e^{k_1 x}$, will be dominated by the growing exponential.) To give G(k) a pole at $-k_1$ the right contour must pass to the left of the pole of P(k) at $-k_1$. Since the left-contour must always be to the left of the right contour, the two contours must be taken as in Figure 7. (Other contour arrangements are possible, e.g., but the solutions so obtained may be represented as linear combinations of the two solutions obtained from the contours of Figure 6 and 7.

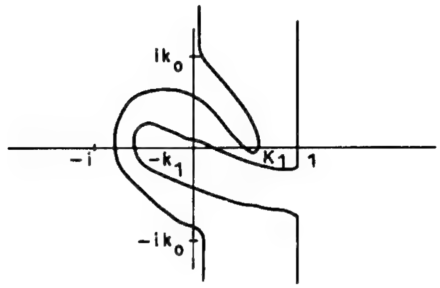


Figure 13.

Deforming the contours of Figure 7 so as to permit simplification of the integrals gives this form:

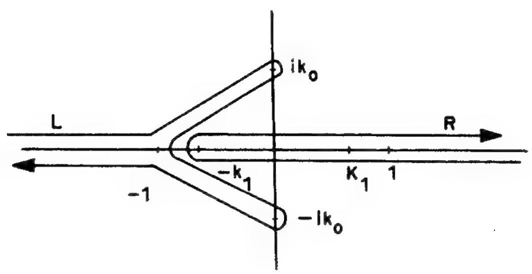


Figure 14.

Taking as before:

$$\log P_L(k) = -\frac{1}{2\pi i}\int_L \frac{dk'}{k' - k}\log P(k') = \log F(k) + \text{constant}$$

$$\log P_R(k) = \frac{1}{2\pi i}\int_R \frac{dk'}{k' - k}\log P(k') = \log G(k) + \text{constant}$$

The integral, $\log P_R(k)$, may be broken up into pieces which have been evaluated previously.

$$\log P_R(k) = \frac{1}{2\pi i}\int_R \frac{dk'}{k' - k}\log\left(\frac{c}{2k'}\log\frac{1 + k'}{1 - k'} - 1\right) - \frac{1}{2\pi i}\int_R \frac{dk'}{k' - k}\log\left(1 - \frac{c'}{2k'}\log\frac{1 + k'}{1 - k'}\right)$$

$$= \frac{1}{\pi}\int_0^1 \frac{ds}{s(1 - ks)}\, T_c - \frac{1}{2\pi i}\int_{-k_1}^{\infty}\frac{dk'}{k' - k}(-2\pi i)$$

$$- \frac{1}{2\pi i}\int_{R\ \text{decaying}} \frac{dk'}{k' - k}\log\left(1 - \frac{c'}{2k'}\log\frac{1 + k'}{1 - k'}\right)$$

The last term has been evaluated in getting $\log P_R(k)$ for the decaying solution.

$$\log P_R(k) = \frac{1}{\pi} \int_0^1 \frac{ds}{s(1 - ks)} T_c + \int_0^{-1/k_1} \frac{ds}{s(1 - ks)} + \int_0^{1/k_1} \frac{ds}{s(1 - ks)} - \frac{1}{\pi} \int_0^1 \frac{ds}{s(1 - ks)} T_{c'}$$

$$= \frac{k}{\pi} \int_0^1 \frac{ds}{1 - ks} (T_c - T_{c'}) + \frac{1}{\pi} \int_0^1 \frac{ds}{s} (T_c - T_{c'}) + 2 \int_0 \frac{ds}{s} - \log (k^2 - k_1^2).$$

$$\log G(k) = \frac{k}{\pi} \int_0^1 \frac{ds}{1 - ks} (T_c - T_{c'}) - \log (k_1^2 - k^2) + \log B' \tag{2.24}$$

It may be observed that the G(k) here obtained differs by a factor of $\frac{B'}{k_1(k + k_1)B}$ from the G(k) previously obtained. Since the ratio of F(k) to G(k) is the same, the two F(k)'s must differ by the same factor. We may therefore write $\log F(k)$ immediately

$$\log F(k) = \frac{k}{\pi} \int_0^1 \frac{ds}{1 + ks} (T_c - T_{c'}) + \log \frac{B' k_o^2 (1 - c')}{k_1^2 (k^2 + k_o^2)(c - 1)}$$

B' is again to be evaluated to give the asymptotic sine solution unit amplitude.

$$f(x) = \sin k_o(x + x_1) + h(x),\ x > o,\ h(x) \to o \text{ as } x \to \infty \tag{2.25}$$

$$F(k) = \frac{1}{2i} \left(\frac{e^{ik_o x_1}}{k - ik_o} - \frac{e^{-ik_o x_1}}{k + ik_o} \right) + H(k).$$

$$\lim_{\epsilon \to o} \left[\log F(ik_o + \epsilon) - \log F(- ik_o + \epsilon) \right] = \log (- 1) + 2ik_o x_1,$$

$$= \frac{2ik_o}{\pi} \int_0^1 \frac{ds}{1 + k_o^2 s^2} (T_c - T_{c'}) + \log (- 1)$$

$$x_1 = \frac{1}{\pi} \int_0^1 \frac{ds}{1 + k_o^2 s^2} (T_c - T_{c'}) \quad (x_1 < 0 \text{ since } T_c < T_{c'} \text{ for } 0 < s < 1) \tag{2.26}$$

$$\lim_{\epsilon \to o} \left[\log F(ik_o + \epsilon) + \log F(- ik_o + \epsilon) + 2 \log \epsilon \right] = - 2 \log 2$$

$$= \frac{2k_o^2}{\pi} \int_0^1 \frac{s\, ds}{1 + k_o^2 s^2} (T_c - T_{c'}) + 2 \log \frac{B' k_o^2 (1 - c')}{k_1^2 (c - 1)} - 2 \log (2k_o)$$

$$\log B' = \log \frac{k_1^2 (c - 1)}{k_o (1 - c')} - \frac{k_o^2}{\pi} \int_0^1 \frac{s ds}{1 + k_o^2 s^2} (T_c - T_{c'})$$

$$= \log \frac{k_1^2 (c - 1)}{k_o (1 - c')} - \frac{1}{2} \log \frac{2(c - 1) k_1^2 (1 - c'/c)}{[1 - c/(1 + k_o^2)] (k_1^2 + k_o^2)(1 - c')} \qquad \text{(cf. 2.19)}$$

$$= \frac{1}{2}\log\frac{k_1^2(c-1)\left[1-c/(1+k_o^2)\right](k_1^2+k_o^2)}{2k_o^2(1-c')(1-c'/c)}$$

$$\log F(k) = \frac{k}{\pi}\int_o^1 \frac{ds}{1+ks}(T_c - T_{c'}) + \log B' + \log\frac{k_o^2(1-c')}{k_1^2(k^2+k_o^2)(c-1)}$$

$$= \frac{k}{\pi}\int_o^1 \frac{ds}{1+ks}(T_c - T_{c'}) + \frac{1}{2}\log\frac{k_o^2(1-c')\left[1-c/(1+k_o^2)\right](k_1^2+k_o^2)}{2k_1^2(c-1)(k^2+k_o^2)^2(1-c'/c)}$$

$$F(k) = \frac{k_o\sqrt{k_1^2+k_o^2}}{k_1(k^2+k_o^2)}\sqrt{\frac{(1-c')\left[1-c/(1+k_o^2)\right]}{2(c-1)(1-c'/c)}}\; e^{\frac{k}{\pi}\int_o^1 \frac{ds}{1+ks}(T_c - T_{c'})}$$

$$H(k) = \frac{k_o\sqrt{k_1^2+k_o^2}}{k_1(k^2+k_o^2)}\sqrt{\frac{(1-c')\left[1-c/(1+k_o^2)\right]}{2(c-1)(1-c'/c)}}\; e^{\frac{k}{\pi}\int_o^1 \frac{ds}{1+ks}(T_c - T_{c'})}$$

$$- \frac{k\sin k_o x_1 + k_o\cos k_o x_1}{k^2+k_o^2}$$

$$h(x) = \frac{1}{2\pi i}\int_{-i\infty+\delta}^{i\infty+\delta} dk\, H(k)\, e^{kx} = \frac{1}{2\pi i}\int_{-L''} dk\, F(k)\, e^{kx}, \quad \text{(cf. Fig 12)},$$

since H(k) is regular at $\pm ik_o$ and F(k) - F(k) - H(k) is single-valued across the $-\infty \longrightarrow -1$ cut.

$$h(x) = \frac{1}{2\pi i}\int_{-\infty}^{-1} dk\, e^{kx}\, \frac{D(c-1)k_1^2\, e^{\frac{k}{\pi}\int_o^1 \frac{ds}{1-ks}(T_c - T_{c'})}}{k_o^2(k_1^2-k^2)(1-c')}\left\{\frac{1-\frac{c'}{2k}\left(\log\left|\frac{1+k}{1-k}\right| - \pi i\right)}{\frac{c}{2k}\left(\log\left|\frac{1+k}{1-k}\right| - \pi i\right)-1} - \frac{1-\frac{c'}{2k}\left(\log\left|\frac{1+k}{1-k}\right| + \pi i\right)}{\frac{c}{2k}\left(\log\left|\frac{1+k}{1-k}\right| + \pi i\right)-1}\right\}$$

where

$$D = \frac{k_o\sqrt{k_1^2+k_1^2}}{k_1}\sqrt{\frac{(1-c')\left[1-c/(1+k_o^2)\right]}{2(c-1)(1-c'/c)}}$$

$$h(x) = \frac{k_1 c\sqrt{k_1^2+k_o^2}}{2k_o}\sqrt{\frac{\left[1-c/(1+k_o^2)\right](c-1)(1-c'/c)}{2(1-c')}}\int_1^\infty \frac{k\,dk}{k^2-k_1^2}\, \frac{e^{-\frac{k}{\pi}\int_o^1 \frac{ds}{1+ks}(T_c - T_{c'})}}{\left(\frac{c}{2}\log\frac{k+1}{k-1} - k\right)^2 + \left(\frac{\pi c}{2}\right)^2}\, e^{-kx}$$

$$\log G(k) = \frac{k}{\pi}\int_o^1 \frac{ds}{1-ks}(T_c - T_{c'}) - \log(k_1^2 - k^2) + \log B'.$$

$$G(k) = \frac{k_1\sqrt{k_1^2+k_o^2}}{k_o(k_1^2 - k^2)}\sqrt{\frac{(c-1)\left[1 - c/(1+k_o^2)\right]}{2(1-c')(1-c'/c)}}\ e^{\frac{k}{\pi}\int_o^1 \frac{ds}{1-ks}(T_c - T_{c'})}$$

$$= \text{, say, } \frac{C}{k_1^2 - k^2}\ e^{\frac{k}{\pi}\int_o^1 \frac{ds}{1-ks}(T_c - T_{c'})}$$

$G(k)$ has simple poles at $\pm k_1$ and a branch point at -1. We will therefore be able to write $g(x)$ as

$$g(x) = Ae^{-k_1x} + Be^{k_1x} + j(x),\ j(x) = 0(e^x) \text{ as } x \to -\infty$$

$$G(k) = \frac{A}{-k-k_1} + \frac{B}{-k+k_1} + J(k),$$

$$A = -\frac{C}{2k_1}\ e^{-\frac{k_1}{\pi}\int_o^1 \frac{ds}{1+k_1s}(T_c - T_{c'})}$$

$$B = +\frac{C}{2k_1}\ e^{\frac{k_1}{\pi}\int_o^1 \frac{ds}{1-k_1s}(T_c - T_{c'})}$$

$$e^{\pm\frac{k_1}{\pi}\int_o^1 \frac{ds}{1\mp k_1s}(T_c - T_{c'})} = e^{\frac{k_1^2}{\pi}\int_o^1 \frac{s\,ds}{1-k_1^2s^2}(T_c - T_{c'}) \pm \frac{k_1}{\pi}\int_o^1 \frac{ds}{1-k_1^2s^2}(T_c - T_{c'})}$$

$$J(k) = \frac{\sqrt{c\left[1 - c/(2+k_o^2)\right]}}{k_1^2 - k^2}\left\{\frac{k_1\sqrt{k_1^2+k_o^2}}{k_o}\sqrt{\frac{C-1}{2(1-c')(c-c')}}\ e^{\frac{k}{\pi}\int_o^1 \frac{ds}{1-ks}(T_c - T_{c'})} - \frac{k\sinh k_1x_2 + k_1\cosh k_1}{\sqrt{c'\left[c'/(1-k_1^2)-2\right]}}\right\}$$

$$g(x) = \frac{C}{k_1}\ e^{\frac{k_1^2}{\pi}\int_o^1 \frac{s\,ds}{1-k_1^2s^2}(T_c - T_{c'})}\sinh k_1(x+x_2) + j(x),$$

where
$$x_2 = \frac{1}{\pi}\int_o^1 \frac{ds}{1-k_1^2s^2}(T_c - T_{c'}),\ (x_1 < x_2 < 0)$$

$$\frac{k_1^2}{\pi}\int_o^1 \frac{s\,ds}{1-k_1^2s^2}(T_c - T_{c'}) = -\frac{1}{2}\log\frac{(k_1^2+k_o^2)(c-1)\left[c'/(1-k_1^2)-1\right]}{k_o^2(c/c'-1)2(1-c')} \qquad \text{(cf. 2.21)}$$

$$g(x) = \sqrt{\frac{c\left[1 - c/(1+k_o^2)\right]}{c'\left[c'/(1-k_1^2)-1\right]}}\ \sinh k_1(x+x_2) + j(x) \qquad (2.27)$$

$$j(x) = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} dk\, e^{kx}\left\{\frac{C}{k_1^2-k^2}\ e^{\frac{k}{\pi}\int_o^1 \frac{ds}{1-ks}(T_c - T_{c'})} - \frac{A}{-k-k_1} - \frac{B}{-k+k_1}\right\}$$

$$= \frac{C}{2\pi i} \int_{R'} \frac{dk e^{kx}}{k_1^2 - k^2} e^{\frac{k}{\pi}\int_0^1 \frac{ds}{2 + ks} (T_c - T_{c'}) \frac{k_o^2(k_1^2 - k^2)(1 - c')}{(k^2 + k_o^2)(c - 1)k_1^2} \frac{\frac{c}{2k}\log\frac{1+k}{1-k} - 1}{1 - \frac{c'}{2k}\log\frac{1+k}{1-k}}}$$

$$j(x) = \frac{(c - c')\, C\, k_o^2(1 - c')}{2k_1^2(c - 1)} \int_1^\infty \frac{kdk\, e^{kx}\, e^{\int_0^1 \frac{ds}{1 + ks}(T_c - T_{c'})}}{(k^2 + k_o^2)\left[\left(k - \frac{c'}{2}\log\frac{k+1}{k-1}\right)^2 + \left(\frac{\pi c'}{2}\right)^2\right]}$$

$$j(x) = \frac{k_o c\sqrt{k_1^2 + k_o^2}}{2k_1} \sqrt{\frac{(1 - c')(1 - c'/c)\left[1 - c/(1 + k_o^2)\right]}{2(c - 1)}}$$

$$\int_1^\infty \frac{k\, dk\, e^{\int_0^1 \frac{ds}{1 + ks}(T_c - T_{c'})}}{(k^2 + k_o^2)\left[\left(k - \frac{c'}{2}\log\frac{k+1}{k-1}\right)^2 + \left(\frac{\pi c'}{2}\right)^2\right]} e^{kx}, \quad (x < 0)$$

We now have two solutions whose asymptotic forms are:

$$\sin k_o\left(x + x_1 + \frac{1}{k_o}\tan^{-1}\frac{k_o}{k_1}\right) \leftrightarrow \frac{k_1\sqrt{c\left[1 - c/(1 + k_o^2)\right]}}{\sqrt{k_1^2 + k_o^2}\sqrt{c'\left[c'/(1 - k_1^2) - 1\right]}}\, e^{k_1(x + x_2)}$$

(cf. equations 2.17, 2.18, 2.22)

$$\sin k_o(x + x_1) \leftrightarrow \frac{\sqrt{c\left[1 - c/(1 + k_o^2)\right]}}{\sqrt{c'\left[c'/(1 - k_1^2) - 1\right]}} \sinh k_1(x + x_2)$$

(cf. equations 2.25, 2.26, 2.27)

We introduce the notation,

$$\beta \equiv \sqrt{c\left[1 - c/(1 + k_o^2)\right]}$$

$$\beta' \equiv \sqrt{c'\left[c'/(1 - k_1^2) - 1\right]}$$

$$n_o(x) \leftrightarrow \frac{\sqrt{k_1^2 + k_o^2}}{k_1\beta} \sin k_o\left(x + x_1 + \frac{1}{k_o}\tan^{-1}\frac{k_o}{k_1}\right) \leftrightarrow \frac{e^{k_1(x + x_2)}}{\beta}$$

$$n_1(x) \leftrightarrow \frac{\sin k_o(x + x_1)}{\beta} \leftrightarrow \frac{\sinh k_1(x + x_2)}{\beta'}$$

$n_o(x)$ is $\frac{\sqrt{k_1^2 + k_o^2}}{k_1\beta}$ times the "decaying solution" first obtained (2.14 to 2.23). $n_1(x)$ is $1/\beta$ times the "growing solution" next obtained (2.24 to 2.27). Subtracting $k_1 n_1(x)$ from $k_1 n_o(x)$ gives

$$n_2(x) = k_1 n_o(x) - k_1 n_1(x)$$

$$\longleftrightarrow \frac{\sqrt{k_1^2 + k_0^2}}{\beta}\left(\sin k_0(x + x_1)\frac{k_1}{\sqrt{k_1^2 + k_0^2}} + \cos k_0(x + x_1)\frac{k_0}{\sqrt{k_1^2 + k_0^2}}\right)$$

$$- \frac{k_1}{\beta}\sin k_0(x + x_1)$$

$$= \frac{k_0}{\beta}\cos k_0(x + x_1) \longleftrightarrow \frac{k_1}{\beta'}\cosh k_1(x + x_2)$$

If we now subtract $n_1(x)$ from $\frac{n_2(x)}{k_1}$ we get

$$n_3(x) = \frac{n_2(x)}{k_1} - n_1(x) \longleftrightarrow \frac{1}{\beta}\left[\cos k_0(x + x_1)\cdot\frac{k_0}{k_1} - \sin k_0(x + x_1)\right]$$

$$= -\frac{\sqrt{k_1^2 + k_0^2}}{k_1\beta}\sin k_0\left(x + x_1 - \frac{1}{k_0}\tan^{-1}\frac{k_0}{k_1}\right)$$

$$\longleftrightarrow \frac{1}{\beta'}e^{-k_1(x + x_2)}$$

We now have two simple pairs of linearly independent solutions, $n(x)$ and $n_2(x)$; $n_0(x)$ and $n_3(x)$. For any one of these four solutions, hence also for any other solution made from them as linear combinations, the asymptotic solutions on the two sides and the derivatives of the asymptotic solutions have a constant ratio when evaluated at $x = -x_1$ and $x = -x_2$ for the core and tamper solutions respectively.

$$\frac{\text{asymptotic core solution } (x = -x_1)}{\text{asymptotic tamper solution } (x = -x_2)} = \frac{-k_0\beta'}{k_1\beta} = \frac{\text{derivative of asymptotic core solution } (x = -x_1)}{\text{derivative of asymptotic tamper solution } (x = -x_2)}$$

the points, $-x_1$ and $-x_2$, are both on the core side of the interface, $-x_2$ being the farther from the interface. This property leads to the following recipe:

In each medium the asymptotic solution is one of the family of solutions of the equation: $(\Delta + k^2)\,n(x) = 0$, $\frac{k}{\tan^{-1}k} = c$ (k may be either real or imaginary). Each of the two asymptotic solutions to be joined at an interface is examined at its "fiducial point", distant Δx from the interface on the side of greater c.

$$\Delta x = \frac{1}{\pi}\int_0^1 \frac{ds}{1 - k^2s^2}\left|T_c - T_{c'}\right| \; *$$

(The Δx for each solution uses its own k which may be either real or imaginary.) The two asymptotic solutions, each at its own fiducial point, have equal logarithmic derivatives. The magnitudes of the two solutions, evaluated at their fiducial points, have the same ratio as their values of the quantity,

$$\frac{k}{\beta} = \sqrt{\frac{k^2}{c\,[1 - c/(1 + k^2)]}} = \sqrt{\frac{k^2}{c\,[c/(1 - K^2) - 1]}} \quad (\text{for } K = ik)$$

---

* See Table 3, which gives $c \cdot \Delta X$.

This recipe paraphrases the connection-formulae given above identifying the two asymptotic solutions on the two-sides of an interface. It differs from a simple diffusion theoretic boundary condition connecting the asymptotic solutions only in so far as

1) Δx differs from 0

2) $\frac{k}{\beta}$ differs from a constant

This recipe connects only the asymptotic solutions. Detailed features of the solutions may be gotten from Table 1.

Symbols used in Table 1.

$$T_c = \tan^{-1}\left[\frac{n/2}{\tanh^{-1}s - 1/cs}\right], \quad T_c(0) = \pi, \; T_c(1) = 0$$

In untamped solution

$$x_o = \frac{1}{\pi}\int_o^1 \frac{ds}{1 + k_o^2 s^2}\, T_c\,, \quad \frac{k_o}{\tan^{-1}k_o} = c, \; \beta = \sqrt{c\left[1 - c/(1 + k_o^2)\right]}, \; c > 1$$

$$x_o = \frac{1}{\pi}\int_o^1 \frac{ds}{1 - k_1^2 s^2}\, T_c\,, \quad \frac{k_1}{\tanh^{-1}k_1} = c, \; \beta' = \sqrt{c\left[c/(1 - k_1^2) - 1\right]}, \; c < 1.$$

In tamped (two-medium) solutions the formulae have been written for the case $c > 1$, $c' < 1$. Other cases follow by analytic extensions.

$$\frac{k_o}{\tan^{-1}k_o} = c$$

$$k_2 = \sqrt{k_o^2 + k_1^2}$$

$$\frac{k_1}{\tanh^{-1}k_1} = c'$$

$$\beta = \sqrt{c\left[1 - c/(1 + k_o^2)\right]}$$

$$\beta' = \sqrt{c'\left[c'/(1 - k_1^2) - 1\right]}$$

$$x_1 = \frac{1}{\pi}\int_o^1 \frac{ds}{1 + k_o^2 s^2}\,(T_c - T_{c'})$$

for $(x_2 < x_1 < 0)$

$$x_2 = \frac{1}{\pi}\int_o^1 \frac{ds}{1 - k_1^2 s^2}\,(T_c - T_{c'})$$

Each of the four solutions is presented as an asymptotic solution in each medium (sinusoidal or hyperbolic) to which is added a discrepancy term $\left(h(x) \text{ for } x>0,\ j(x) \text{ for } x<0\right)$. This discrepancy term may be of either sign.

APPENDIX I

ACCURACY OF TWO-BOUNDARY APPROXIMATION

To estimate the error introduced by neglecting the interaction of two boundaries we determine the effect of this neglect in the untamped sphere problem as a first order perturbation. The fundamental eigenvalue, c, of the equation,

$$n(x) = c\int_{-a}^{a} dx'\, n(x')\frac{1}{2}E(|x - x'|),\ n(-x) = -n(x). \qquad \text{(i)}$$

we write as $c = c_0/(1 + \epsilon) + 0(\epsilon^2)$, where $a = \frac{\pi}{k(c_0)} - x_0(c_0)$.

The integral operator

$$\int_{-\infty}^{\infty} dx'\ \frac{c}{2}\ E(|x - x'|)$$

we denote by Λ .

$$\begin{aligned} \text{Write } R \equiv R(x) &= 0 \text{ for } x < -a \\ &= 1 \text{ for } x > -a \\ L \equiv L(x) &= 0 \text{ for } x > a \\ &= 1 \text{ for } x < a \end{aligned}$$

Equation (i) becomes

$$\begin{gathered} (1 + \epsilon - \Lambda RL)\, n(x) = 0, \text{ valid for } -a \leq x \leq a \\ n(x) = n_0(x) + n_1(x) \\ n_0(x) = n_R(x) + n_L(x) - \sin k_0x \end{gathered} \qquad \text{(ii)}$$

where $n_R(x)$ and $n_L(x)$ are the exact one-boundary solutions satisfying

$$(1 - \Lambda R)n_R = (1 - \Lambda L)n_L = 0$$

$$n_R(x) = R \sin k_0x + h_R(x)$$

$$n_L(x) = L \sin k_0x + h_L(x)$$

Then

$$\begin{aligned} (1 + \epsilon - \Lambda RL)n_1 &= (\Lambda RL - 1 - \epsilon)n_0 = (\Lambda RL - 1)(n_R + n_L - \sin k_0x) - \epsilon n_0 \\ &= \left[\Lambda R - 1 - \Lambda R(1 - L)\right]n_R + \left[\Lambda L - 1 - \Lambda L(1 - R)\right]n_L \\ &\qquad - \left[\Lambda - 1 + \Lambda(RL - 1)\right]\sin k_0x - \epsilon n_0 \\ &= -\Lambda\left[(1 - L)n_R + (1 - R)n_L + (RL - 1)\sin k_0x\right] - \epsilon n_0 \end{aligned}$$

$$= - \Lambda \left[(1 - L)h_R + (1 - R(h_L + (R - RL + L - RL + RL - 1)\sin k_o x \right] - \epsilon n_o$$

$$= - \Lambda \left[(1 - L)h_R + (1 + R(h_L \right] - \epsilon n_o \qquad \text{(iii)}$$

Since n_1 must be finite, the right side of (iii) must contain no component, n(x), satisfying (ii). Neglecting terms of order ϵ^2 we have

$$\int_{-a}^{a} dx\, n(x) \left\{ \Lambda \left[(1 - L)h_R + (1 - R)h_L \right] + \epsilon\, n_o \right\} = 0$$

$$\epsilon \int_{-a}^{a} dx\, n_o^2(x) = - \int_{-\infty}^{\infty} dx\, RL\, n(x)\, \Lambda \left[(1 - L)h_R + (1 - R)h_L \right]$$

$$= - \int_{-\infty}^{\infty} dx \left[(1 - L)h_R + (1 - R)h_L \right] \Lambda\, RL\, n(x)$$

$$= - \int_{-\infty}^{\infty} dx \left[(1 - L)h_R + (1 - R)h_L \right] n(x) \qquad \text{(iv)}$$

The left term of (iv) is roughly 2 a. The right term is minus twice the integral of the discrepancy term, h_R (>0) starting from a point distant 2a from its boundary, with n(x) beyond x = a. The character of n(x) in this region may be determined by taking c ' = 0 in the decaying two-medium solution. Its value at the surface is

$$\frac{\beta}{\sqrt{2(c - 0)}} = \sqrt{\frac{1 - c/(1 + k_o^2)}{2}}$$

The right term of (iv) will be approximately (-2) x $\frac{1 - c/(1 + k_o^2)}{2}$ · h(2a) divided by their combined decay-rate, about 3-4.

For a tamped sphere we proceed in a similar way:

$$\left\{ 1 + \epsilon - \Lambda \left[RL + (1 - RL) \frac{c'}{c} \right] \right\} n(x) = 0$$

$$n = n_o + n_1 = n_R + n_L - \sin k_o x + n_1$$

$$\left\{ 1 - \Lambda \left[R + (1 - R) \frac{c'}{c} \right] \right\} n_R = \left\{ 1 - \Lambda \left[L + (1 - L) \frac{c'}{c} \right] \right\} n_L = 0$$

$$\left\{ 1 + \epsilon - \Lambda \left[\frac{c - c'}{c} RL + \frac{c'}{c} \right] \right\} n_1 = \left\{ \Lambda \left[\frac{c - c'}{c} RL + \frac{c'}{c} \right] - 1 \right\} \cdot (n_R + n_L - \sin k_o x) - \epsilon\, n_o$$

$$= \left\{ \Lambda \left[R + (1 - R) \frac{c'}{c} \right] - 1 \right\} n_R + \Lambda\, R(1 - L) \left(\frac{c'}{c} - 1 \right) n_R$$

$$+ \left\{ \Lambda \left[L + (1 - L) \frac{c'}{c} \right] - 1 \right\} n_L + \Lambda\, L(1 - R) \left(\frac{c'}{c} - 1 \right) n_L$$

$$+ \left\{ 1 - \Lambda \left[\frac{c - c'}{2} RL + \frac{c'}{c} \right] \right\} \sin k_o x - \epsilon\, n_o$$

$$= - \Lambda\, (1 - L) \left(\frac{c - c'}{c} \right) (R \sin k_o x + h_R + g_R)$$

$$- \Lambda (1 - R) \left(\frac{c - c'}{c}\right) (L \sin k_0 x + h_L + g_L)$$

$$+ \left\{1 - \Lambda \left(\frac{c - c'}{c}\right) RL - \frac{c'}{c} \Lambda \right\} \sin k_0 x - \epsilon n_0$$

$$= (1 - \Lambda) \sin k_0 x - \frac{c - c'}{c} \Lambda \left\{(1 - L)h_R + (1 - R)h_L\right\} - \epsilon n_0$$

$$= - \left(1 - \frac{c'}{c}\right) \Lambda \left\{(1 - L) h_R + (1 - R)h_L\right\} - \epsilon n_0$$

Hence as before:

$$\epsilon \sim - \frac{2}{a} \left(1 - \frac{c'}{c}\right) \int dx\, n_0(x)\, \Lambda \left\{(1 - L) h_R + (1 - R)h_L\right\}$$

$$\sim - \frac{2}{a} \left(1 - \frac{c'}{c}\right) \int_a^\infty dx\, n_0(x)\, h_R(x)$$

Estimating this integral in the same way as before gives, for example, for $c = 2.0$, $c' = 1.0$,

$$\epsilon - \frac{2}{.72} \times \frac{.5 \times .71 \times .003}{2} \sim .0015$$

For $c' = 1$ and various values of c, we obtain the estimates:

c	ϵ	% in critical radius
1.5	.0002	.09
2.0	.0015	.53
2.5	.003	1.0
3.0	.005	1.3
∞	.02	2.0

The chief factor making these errors small is the rapid decay of h(x). Taking the untamped-solution values as typical (they will actually be somewhat too large) it would appear that ϵ will exceed .01 only for core diameters or tamper thicknesses considerably less than one mean free path.

Comparison with variation theory results gives about 0.3 as the limiting thickness for 1 per cent accuracy. (cf. Comparison of variation theory and end point results for tamped spheres, LADC - 77)

APPENDIX II

SOLUTION OF THE INHOMOGENEOUS WIENER-HOPF EQUATION

The Wiener-Hopf technique was shown by E. Reissner (Journal of Mathematics and Physics, Vol. XX (1941), pp 219-223) to permit extension to the inhomogeneous problem. We here treat only the one medium problem with the inhomogeneous term confined to $x \geqq 0$. The extension to the two-medium problem with an unrestricted inhomogeneous term is immediately obvious. The equation we wish to solve is:

$$n(x) = \int_0^\infty dx'\, n(x')\, K(x - x') + f_1(x) \qquad \text{(a)}$$

where $f_1(x)$ is known and vanishes for $x < 0$. The Laplace transform of (a), with the notation used previously is,

$$G(k) = F(k)\,(\underline{K}(k) - 1) + F_1(k) = F(k)\,P(k) + F_1(k), \tag{b}$$

$$F_1(k) = \int_0^\infty dx\, f_1(x)\, e^{-kx}$$

The solution of the corresponding homogeneous equation will be denoted by a subscript 0.

$$G_0(k) = F_0(k)\,P(k)$$

$$P(k) = G_0(k)/F_0(k)$$

We define $\underline{F}(k)$ such that

$$F(k) = F_0(k)\,\underline{F}(k)$$

This introduces no singularities in $\underline{F}(k)$ in the right half-plane since $F_0(k)$ had no roots in the right half-plane. Then (b) becomes,

$$F(k)\,P(k) = \underline{F}(k)\,F_0(k)\left(\frac{G_0(k)}{F_0(k)}\right) = \underline{F}(k)\,G_0(k) = G(k) - F_1(k)$$

Thus $-F_1(k)$ is the right-analytic component of $\underline{F}(k)\,G_0(k)$, which we may write as

$$\left[\underline{F}(k)\,G_0(k)\right]_R = \frac{1}{2\pi i}\int_L \frac{dk'}{k' - k}\,\underline{F}(k')\,G_0(k'),$$

where the contour L lies to the left of k and of the singularities of $G_0(k)$ (which are entirely in the right half-plane) and to the right of the singularities of $\underline{F}(k)$ (in the left half-plane).

$$\left[\underline{F}(k)\,G_0(k)\right]_R = -\,F_1(k) \tag{c}$$

Making use of the fact that $\frac{1}{G_0(k)}$ as well as $G_0(k)$ is analytic in the left half-plane we can show that equation c is satisfied by

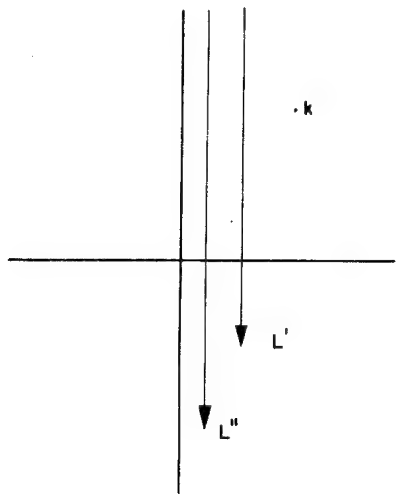
$$\underline{F}(k) = -\left[F_1(k)\,\frac{1}{G_0(k)}\right]_R \tag{d}$$

since

$$\left[G_0(k)\underline{F}(k)\right]_R = -\left[G_0(k)\left[F_1(k)\,\frac{1}{G_0(k)}\right]_R\right]_R$$

$$= \frac{-1}{(2\pi i)^2}\int_{L'} \frac{dk'}{k' - k}\,G_0(k')\int_{L''} \frac{dk''}{k'' - k'}\,\frac{F_1(k'')}{G_0(k'')}$$

$$\left[G_0(k)\,\underline{F}(k)\right]_R = -\frac{1}{(2\pi i)^2}\int_{L''} dk''\,\frac{F_1(k'')}{G_0(k'')}\int_{L'} dk'\,G_0(k')\,\frac{1}{k'' - k}\left(\frac{1}{k' - k} + \frac{1}{k'' - k'}\right)$$



Displacing the contour L′ to the left of L″ picks up a residue at $k' = k''$. The remaining k′ integral vanishes as it may be displaced indefinitely to the left, in which direction the integrand decays as $\frac{1}{|k'|^2}$. This leaves:

$$\left[G_o(k)\,\underline{F}(k)\right]_R = -\frac{1}{(2\pi i)^2}\int_{L''} dk''\,\frac{F_1(k'')}{G_o(k'')}\left(\frac{2\pi i}{k''-k}\cdot G_o(k'')\right)$$

$$= -\left[F_1(k)\right]_R = -F_1(k)$$

The particular integral of equation a has therefore the Laplace transform

$$F(k) = -F_o(k)\left[\frac{F_1(k)}{G_o(k)}\right]_R$$

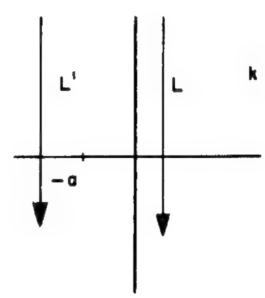
To this may be added any multiple of the homogeneous solution, $F_o(k)$.

To extend this method of solution to the two-medium problem requires only the replacement of equation a by the corresponding two-medium equation. This leaves the form of equation b and the rest of the solution unchanged. To treat an inhomogeneous term existing for both $x > 0$ and $x < 0$ it suffices to break up the inhomogeneous term into a right and a left side part and treat each separately as above.

A particularly simple special case of the untamped inhomogeneous equation is that of the albedo problem—

$$f_1(x) = e^{-\alpha x} \quad \alpha > 0.$$

$$F_1(k) = \frac{1}{k+\alpha}$$



Then

$$\left[\frac{F_1(k)}{G_0(k)}\right]_R = \frac{1}{2\pi i}\int_L \frac{dk}{k' - k}\;\frac{1}{(k' + \alpha)G_0(k')}$$

$$= \frac{1}{G_0(-\alpha)(k+\alpha)} + \frac{1}{2\pi i}\int_{L'} \frac{dk'}{(k' - k)(k' + \alpha)G_0(k')}$$

In the second term the contour L' may be displaced indefinitely to the left. Its integrand may be written as

$$\frac{\text{Const.}}{k'} + 0\left(\frac{1}{k'^{\,2}}\right)$$

Thus the k-dependent part of the integral vanishes. The constant part represents an admixture of the homogeneous solution to $F_1(k)$ and therefore may be disregarded. The general solution is therefore

$$F(k) = -F_0(k)\left(\left[\frac{F_1(k)}{G_0(k)}\right]_R + A\right) = -F_0(k)\left(\frac{1}{G_0(-\alpha)(k+\alpha)} + A\right).$$

In an albedo problem c will be ≤ 1 and A should be chosen to make n(x) finite for all x, hence F(k) regular at $k = +k_1$, despite the pole of $F_0(k)$. Thus

$$A = -\frac{1}{G_0(-\alpha)(k_1 + \alpha)}$$

$$F(k) = \frac{(k - k_1)F_0(k)}{(k+\alpha)(k_1 + \alpha)G_0(-\alpha)}$$

The density of emergent neutrons in the albedo problem as a function of μ, the cosine of the angle of emergence, is

$$N(\mu) = c\int_0^\infty dx\; n(x)e^{-x/\mu}$$

$$= cF\,\frac{1}{\mu}$$

and is therefore given directly by the solution F(k).

TABI

Solution	$f(x) - h(x)$ Asymptotic core solution	$g(x) - j(x)$ Asymptotic tamper solution	$n(0) = f(0) = g(0)$ Value of solution at interface	$-H(0) = -\int_0^\infty h(x)\,dx$ Area of discrepancy term in core	$-\frac{H'(0)}{H(0)} = \frac{\int_0^\infty x\,h(x)\,dx}{\int_0^\infty h(x)\,dx}$ Mean length of discref in core
$c>1$ UNTAMPED	$\sin K_0(x+x_0)$		$\frac{\beta}{\sqrt{2c}}$	$-\frac{1}{K_0}\left[\frac{\beta}{\sqrt{2c(c-1)}} - \cos K_0 x_0\right]$	$\frac{1}{H(0)K_0^2}\Big[\sin K_0 x_0$ $- K_0 \frac{\beta}{\sqrt{2c(c-1)}} \frac{1}{\pi}\int_0^1 d$
CASE $c<1$	$\sinh K_1(x+x_0)$		$\frac{\beta'}{\sqrt{2c}}$	$\frac{1}{K_1}\left[\frac{\beta'}{\sqrt{2c(1-c)}} - \cosh K_1 x_0\right]$	$\frac{-1}{H(0)K_1^2}\Big[\sinh K_1 x_0$ $- K_1 \frac{\beta'}{\sqrt{2c(1-c)}} \frac{1}{\pi}\int_0^1 d$
n_0	$\frac{K_2}{K_1\beta}\sin K_0(x+x_0)$ $= \frac{1}{\beta}\sin K_0(x+x_1)$ $+ \frac{K_0}{K_1\beta}\cos K_0(x+x_1)$	$\frac{1}{\beta'} e^{K_1(x+x_2)}$	$\frac{K_2}{K_1\sqrt{2(c-c')}}$	$\frac{K_2}{K_1K_0\beta}\left[\cos K_0 x_0 - \beta\sqrt{\frac{(1-c')}{2(c-1)(c-c')}}\right]$	$\frac{K_2}{(-H(0))K_1K_0^2}\Big[\frac{K_0}{K_1}\sqrt{\frac{(1-c')}{2(c-1)(c-c')}}\Big(1+$ $-\frac{1}{\beta}\sin K_0\Big(x_1 + \frac{1}{K_0}\tan^{-1}$
n_3 $= \frac{n_2}{K_1} - n_1$	$\frac{K_0}{K_1\beta}\cos K_0(x+x_1)$ $- \frac{1}{\beta}\sin K_0(x+x_1)$ $= \frac{-K_2}{K_1\beta}\sin K_0\left(x+x_1-\frac{1}{K_0}\tan^{-1}\frac{K_2}{K_1}\right)$	$\frac{1}{\beta'} e^{-K_1(x+x_2)}$	$\frac{K_2}{K_1\sqrt{2(c-c')}}$	$\frac{K_2}{K_1K_0\beta}\Big[\beta\sqrt{\frac{(1-c')}{2(c-1)(c-c')}}$ $- \cos K_0\left(x_1 - \frac{1}{K_0}\tan^{-1}\frac{K_2}{K_1}\right)\Big]$	$\frac{K_2}{-H(0)K_0^2K_1}\Big[\frac{K_0}{K_1}\sqrt{\frac{(1-c')}{2(c-1)(c-c')}}\Big(1-\frac{K_1}{\pi}\int_0$ $+\frac{1}{\beta}\sin K_0\Big(x_1 - \frac{1}{K_0}\tan$
n_1	$\frac{1}{\beta}\sin K_0(x+x_1)$	$\frac{1}{\beta'}\sinh K_1(x+x_2)$	0	$-\frac{1}{K_0\beta}\left[\frac{K_2\beta}{K_1}\sqrt{\frac{(1-c')}{2(c-1)(c-c')}} - \cos K_0 x_1\right]$ (negative, i.e $H(0)>0$)	$\frac{1}{K_0^2 H(0)}\Big[\frac{\sin K_0 x_1}{\beta}$ $-\frac{K_0K_2}{K_1}\sqrt{\frac{(1-c')}{2(c-1)(c-c')}}\cdot\frac{1}{\pi}\int_0^1 ds\,($
n_2 $= K_1(n_0 - n_1)$	$\frac{K_0}{\beta}\cos K_0(x+x_1)$	$\frac{K_1}{\beta'}\cosh K_1(x+x_2)$	$\frac{K_2}{\sqrt{2(c-c')}}$	$-\frac{\sin K_0 x_1}{\beta}$ (positive, i.e $H(0)<0$)	$\frac{-1}{\sin K_0 x_1\, K_0}\Big[\frac{\beta K_2}{K_1}\sqrt{\frac{(1-c')}{2(c-1)(c}}$ $- \cos K_0 x_1\Big]$

1082

TABLE-I

$\ldots(0) = \frac{\int_0^\infty x\,h(x)\,dx}{\int_0^\infty h(x)\,dx}$ LENGTH OF DISCREPANCY IN CORE	$J(0) = \int_{-\infty}^0 j(x)\,dx$ AREA OF DISCREPANCY IN TAMPER	$\frac{J'(0)}{J(0)} = \frac{-\int_{-\infty}^0 x\,j(x)\,dx}{\int_{-\infty}^0 j(x)\,dx}$ MEAN LENGTH OF DISCREPANCY IN TAMPER	$h(x)$ DISCREPANCY (NEGATIVE) IN CORE $(x \geq 0)$	$j(x)$ DISCREPANCY (POSITIVE) IN TAMPER $(x \leq 0)$
$\ldots)\kappa_0^2 \left[\sin \kappa_0 x_0 - \kappa_0 \frac{\beta}{\sqrt{2c(c-1)}} \frac{1}{\pi}\int_0^1 ds\, T_c \right]$			$-\frac{\beta}{2\kappa_0}\sqrt{\frac{c(c-1)}{2}}\int_1^\infty \frac{\kappa\, d\kappa\, e^{-\frac{\kappa}{\pi}\int_0^1 \frac{ds}{1+\kappa s}T_c}}{\left(\frac{c}{2}\log\frac{\kappa+1}{\kappa-1}-\kappa\right)^2+\left(\frac{\pi c}{2}\right)^2}\, e^{-\kappa x}$	
$\frac{1}{\ldots)\kappa_1^2}\left[\sinh\kappa_1 x_0 - \kappa_1\frac{\beta'}{\sqrt{2c(1-c)}}\frac{1}{\pi}\int_0^1 ds\,T_c\right]$			$-\frac{\beta'}{2\kappa_1}\sqrt{\frac{c(1-c)}{2}}\int_1^\infty \frac{\kappa\, d\kappa\, e^{-\frac{\kappa}{\pi}\int_0^1 \frac{ds}{1+\kappa s}T_c}}{\left(\frac{c}{2}\log\frac{\kappa+1}{\kappa-1}-\kappa\right)^2+\left(\frac{\pi c}{2}\right)^2}\, e^{-\kappa x}$	
$\ldots\kappa_1\kappa_0^2\left[\frac{\kappa_0}{\kappa_1}\sqrt{\frac{(1-c')}{2(c-1)(c-c')}}\left(1+\frac{\kappa_1}{\pi}\int_0^1 ds(T_c-T_{c'})\right) - \frac{1}{\beta}\sin\kappa_0\left(x_1+\frac{1}{\kappa_0}\tan^{-1}\frac{\kappa_0}{\kappa_1}\right)\right]$	$\frac{\kappa_2}{\kappa_1\kappa_0}\sqrt{\frac{c-1}{2(1-c')(c-c')}} - \frac{1}{\kappa_1\beta'}e^{\kappa_1 x_2}$	$\frac{1}{\kappa_1} + \frac{\frac{1}{\pi}\int_0^1 ds(T_c-T_{c'})}{1-\frac{\kappa_0}{\kappa_2\beta'}\sqrt{\frac{2(1-c')(c-c')}{(c-1)}}\,e^{\kappa_1 x_2}}$	$-\frac{\kappa_2}{2\kappa_0}\sqrt{\frac{(c-1)(c-c')}{2(1-c')}}\int_1^\infty\frac{\kappa\, d\kappa\, e^{-\frac{\kappa}{\pi}\int_0^1\frac{ds}{1+\kappa s}(T_c-T_{c'})}}{(\kappa+\kappa_1)\left[\left(\frac{c}{2}\log\frac{\kappa+1}{\kappa-1}-\kappa\right)^2+\left(\frac{\pi c}{2}\right)^2\right]}\,e^{-\kappa x}$	$\frac{\kappa_0\kappa_2}{2\kappa_1^2}\sqrt{\frac{(1-c')(c-c')}{2(c-1)}}\int_1^\infty\frac{\kappa\, d\kappa(\kappa+\kappa_1)\,e^{\frac{\kappa}{\pi}\int_0^1\frac{ds}{1+\kappa s}(T_c-T_{c'})}}{(\kappa^2+\kappa_0^2)\left[\left(\frac{c'}{2}\log\frac{\kappa+1}{\kappa-1}-\kappa\right)^2+\left(\frac{\pi c'}{2}\right)^2\right]}\,e^{\kappa x}$
$\ldots\kappa\left[\frac{\kappa_0}{\kappa_1}\sqrt{\frac{(1-c')}{2(c-1)(c-c')}}\left(1-\frac{\kappa_1}{\pi}\int_0^1 ds(T_c-T_{c'})\right)+\frac{1}{\beta}\sin\kappa_0\left(x_1-\frac{1}{\kappa_0}\tan^{-1}\frac{\kappa_0}{\kappa_1}\right)\right]$	$\frac{e^{-\kappa_1 x_2}}{\kappa_1\beta'} - \frac{\kappa_2}{\kappa_1\kappa_0}\sqrt{\frac{(c-1)}{2(1-c')(c-c')}}$	$-\frac{1}{\kappa_1}+\frac{\frac{1}{\pi}\int_0^1 ds(T_c-T_{c'})}{1-\frac{\kappa_0}{\kappa_2\beta'}\sqrt{\frac{2(1-c')(c-c')}{(c-1)}}\,e^{-\kappa_1 x_2}}$	$\frac{\kappa_2}{2\kappa_0}\sqrt{\frac{(c-1)(c-c')}{2(1-c')}}\int_1^\infty\frac{\kappa\, d\kappa\, e^{-\frac{\kappa}{\pi}\int_0^1\frac{ds}{1+\kappa s}(T_c-T_{c'})}}{(\kappa-\kappa_1)\left[\left(\frac{c}{2}\log\frac{\kappa+1}{\kappa-1}-\kappa\right)^2+\left(\frac{\pi c}{2}\right)^2\right]}\,e^{-\kappa x}$	$\frac{\kappa_0\kappa_2}{2\kappa_1^2}\sqrt{\frac{(1-c')(c-c')}{2(c-1)}}\int_1^\infty\frac{\kappa\, d\kappa(\kappa-\kappa_1)\,e^{\frac{\kappa}{\pi}\int_0^1\frac{ds}{1+\kappa s}(T_c-T_{c'})}}{(\kappa^2+\kappa_0^2)\left[\left(\frac{c'}{2}\log\frac{\kappa+1}{\kappa-1}-\kappa\right)^2+\left(\frac{\pi c'}{2}\right)^2\right]}\,e^{\kappa x}$
$\ldots J(0)\left[\frac{\sin\kappa_0 x_1}{\beta} \ldots\sqrt{\frac{(1-c')}{2(c-1)(c-c')}}\cdot\frac{1}{\pi}\int_0^1 ds(T_c-T_{c'})\right]$	$\frac{\kappa_2}{\kappa_1\kappa_0}\sqrt{\frac{(c-1)}{2(1-c')(c-c')}} - \frac{\cosh\kappa_1 x_2}{\kappa_1\beta'}$	$\frac{1}{J(0)}\left[\frac{\kappa_2}{\kappa_0\kappa_1}\sqrt{\frac{(c-1)}{2(1-c')(c-c')}}\cdot\frac{1}{\pi}\int_0^1 ds(T_c-T_{c'})-\frac{\sinh\kappa_1 x_2}{\kappa_1^2\beta'}\right]$	$\frac{\kappa_2\kappa_1}{2\kappa}\sqrt{\frac{(c-1)(c-c')}{2(1-c')}}\int_1^\infty\frac{\kappa\, d\kappa\, e^{-\frac{\kappa}{\pi}\int_0^1\frac{ds}{1+\kappa s}(T_c-T_{c'})}}{(\kappa^2-\kappa_1^2)\left[\left(\frac{c}{2}\log\frac{\kappa+1}{\kappa-1}-\kappa\right)^2+\left(\frac{\pi c}{2}\right)^2\right]}\,e^{-\kappa x}$	$\frac{\kappa_0\kappa_2}{2\kappa_1}\sqrt{\frac{(1-c')(c-c')}{2(c-1)}}\int_1^\infty\frac{\kappa\, d\kappa\, e^{\frac{\kappa}{\pi}\int_0^1\frac{ds}{1+\kappa s}(T_c-T_{c'})}}{(\kappa^2+\kappa_0^2)\left[\left(\frac{c'}{2}\log\frac{\kappa+1}{\kappa-1}-\kappa\right)^2+\left(\frac{\pi c'}{2}\right)^2\right]}\,e^{\kappa x}$
$\ldots x_1\,\kappa_0\left[\frac{\beta\kappa_2}{\kappa_1}\sqrt{\frac{(1-c')}{2(c-1)(c-c')}} - \cos\kappa_0 x_1\right]$	$-\frac{\sinh\kappa_1 x_2}{\beta'}$	$\frac{-1}{\sinh\kappa_1 x_2\cdot\kappa_1}\left[\frac{\beta'\kappa_2}{\kappa_0}\sqrt{\frac{(c-1)}{2(1-c')(c-c')}} - \cosh\kappa_1 x_2\right]$	$-\frac{\kappa_2\kappa_1}{2\kappa_0}\sqrt{\frac{(c-1)(c-c')}{2(1-c')}}\int_1^\infty\frac{\kappa^2 d\kappa\, e^{-\frac{\kappa}{\pi}\int_0^1\frac{ds}{1+\kappa s}(T_c-T_{c'})}}{(\kappa^2-\kappa_1^2)\left[\left(\frac{c}{2}\log\frac{\kappa+1}{\kappa-1}-\kappa\right)^2+\left(\frac{\pi c}{2}\right)^2\right]}\,e^{-\kappa x}$	$\frac{\kappa_0\kappa_2}{2\kappa_1}\sqrt{\frac{(1-c')(c-c')}{2(c-1)}}\int_1^\infty\frac{\kappa^2 d\kappa\, e^{\frac{\kappa}{\pi}\int_0^1\frac{ds}{1+\kappa s}(T_c-T_{c'})}}{(\kappa^2+\kappa_0^2)\left[\left(\frac{c'}{2}\log\frac{\kappa+1}{\kappa-1}-\kappa\right)^2+\left(\frac{\pi c'}{2}\right)^2\right]}\,e^{\kappa x}$

Table 2. $\frac{k}{\pi \log(1+k)} \int_0^1 \frac{ds}{1+ks} \; T_c$

k \ s	0	0.2	0.4	0.6	0.8	1.0	1.2	1.4	1.6	$\frac{\log(1+k)}{k}$
0.00	1.00000	.94049	.86583	.80017	.74951	.71045	.67963	.65468	.63408	1.00000
0.20	1.00000	.94262	.87110	.80785	.75865	.72044	.69009	.66541	.64493	.91161
0.40	1.00000	.94439	.87541	.81418	.76623	.72875	.69884	.67442	.65407	.84118
0.60	1.00000	.94588	.87903	.81951	.77264	.73584	.70633	.68215	.66194	.78334
0.80	1.00000	.94719	.88220	.82414	.77823	.74202	.71288	.68893	.66885	.73473
1.00	1.00000	.94833	.88494	.82818	.78314	.74746	.71867	.69493	.67499	.69315
1.50	1.00000	.95069	.89056	.83649	.79323	.75873	.73070	.70746	.68784	.61086
2.00	1.00000	.95256	.89497	.84301	.80120	.76767	.74029	.71748	.69816	.54931
2.50	1.00000	.95408	.89856	.84833	.80773	.77502	.74820	.72579	.70674	.50111
3.00	1.00000	.95537	.90157	.85280	.81323	.78124	.75492	.73287	.71407	.46210
4.00	1.00000	.95746	.90640	.85998	.82213	.79130	.76583	.74439	.72603	.40236
5.00	1.00000	.95912	.91018	.86559	.82906	.79922	.77447	.75353	.73556	.35835
∞	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	0.00000

Table 2. (continued) $\frac{k}{\pi \log(1+k)} \int_0^1 \frac{ds}{1+ks} \; T_c$

k / c	1.6	1.8	2.0	2.2	2.4	2.6	2.8	3.0	∞	$\frac{\log(1+k)}{k}$
0.00	.63408	.61673	.60190	.58907	.57783	.56791	.55906	.55112	.38002	1.00000
0.20	.64493	.62763	.61279	.59991	.58860	.57859	.56965	.56161	.38424	.91161
0.40	.65407	.63683	.62200	.60910	.59775	.58767	.57867	.57055	.38774	.84118
0.60	.66194	.64476	.62996	.61705	.60568	.59557	.58651	.57835	.39073	.78334
0.80	.66885	.65175	.63699	.62409	.61270	.60256	.59347	.58526	.39332	.73473
1.00	.67499	.65797	.64324	.63036	.61896	.60881	.59969	.59158	.39560	.69315
1.50	.68784	.67102	.65642	.64359	.63221	.62204	.61289	.60459	.40032	.61086
2.00	.69816	.68154	.66706	.65431	.64297	.63282	.62365	.61533	.40406	.54931
2.50	.70674	.69031	.67596	.66329	.65200	.64187	.63271	.62439	.40713	.50111
3.00	.71407	.69781	.68358	.67100	.65977	.64967	.64054	.63221	.40973	.46210
4.00	.72603	.71010	.69611	.68370	.67258	.66257	.65348	.64518	.41392	.40236
5.00	.73556	.71992	.70614	.69389	.68289	.67297	.66395	.65569	.41721	.35835
∞	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	0.50000	0.00000

Table 3. $c\,\Delta X(c,c') = \frac{c}{\pi}\int_0^1 \frac{T_{c'}-T_c}{1-k^2s^2}\,ds$

c' \ c	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
0	0.00000	.14611	.21486	.25088	.26949	.27934	.28455	.28732	.28874	.28937	.28954
0.1	-1.00000	0.00000	.15289	.20090	.22591	.24020	.24883	.25435	.25802	.26056	.26236
0.2	-1.00000	- .34617	0.00000	.11416	.16116	.18661	.20229	.21280	.22025	.22578	.23002
0.3	-1.00000	- .47445	- .16128	0.00000	.07723	.11948	.14561	.16326	.17595	.18551	.19298
0.4	-1.00000	- .53520	- .25698	- .09190	0.00000	.05434	.08931	.11350	.13120	.14470	.15537
0.5	-1.00000	- .57007	- .31537	- .15625	- .06033	0.00000	.04046	.06922	.09068	.10731	.12058
0.6	-1.00000	- .59264	- .35418	- .20195	- .10618	- .04344	0.00000	.03162	.05563	.07448	.08971
0.7	-1.00000	- .60840	- .38183	- .23571	- .14156	- .07818	- .03328	0.00000	.02565	.04605	.06269
0.8	-1.00000	- .62024	- .40256	- .26164	- .16951	- .10638	- .06089	- .02669	0.00000	.02143	.03906
0.9	-1.00000	- .62935	- .41871	- .28217	- .19212	- .12964	- .08407	- .04941	- .02211	0.00000	.01831
1.0	-1.00000	- .63661	- .43167	- .29887	- .21080	- .14915	- .10377	- .06896	- .04131	- .01878	0.00000
1.1	-1.00000	- .64256	- .44235	- .31272	- .22649	- .16576	- .12072	- .08594	- .05816	- .03537	- .01628
1.2	-1.00000	- .64754	- .45130	- .32443	- .23988	- .18006	- .13547	- .10084	- .07304	- .05012	- .03083
1.3	-1.00000	- .65176	- .45892	- .33447	- .25144	- .19252	- .14841	- .11402	- .08627	- .06331	- .04393
1.4	-1.00000	- .65541	- .46549	- .34318	- .26154	- .20348	- .15989	- .12576	- .09814	- .07520	- .05577
1.5	-1.00000	- .65857	- .47124	- .35082	- .27045	- .21321	- .17011	- .13629	- .10884	- .08597	- .06654
1.6	-1.00000	- .66136	- .47630	- .35757	- .27838	- .22190	- .17932	- .14581	- .11854	- .09578	- .07638
1.7	-1.00000	- .66383	- .48080	- .36360	- .28548	- .22973	- .18762	- .15444	- .12738	- .10474	- .08541
1.8	-1.00000	- .66604	- .48483	- .36901	- .29188	- .23682	- .19518	- .16233	- .13547	- .11297	- .09373
1.9	-1.00000	- .66803	- .48847	- .37390	- .29768	- .24326	- .20208	- .16955	- .14291	- .12056	- .10142
2.0	-1.00000	- .66983	- .49176	- .37834	- .30297	- .24916	- .20841	- .17619	- .14978	- .12758	- .10856
2.1	-1.00000	- .67147	- .49476	- .38241	- .30782	- .25457	- .21425	- .18232	- .15615	- .13411	- .11519
2.2	-1.00000	- .67297	- .49751	- .38613	- .31228	- .25957	- .21964	- .18802	- .16206	- .14019	- .12139
2.3	-1.00000	- .67435	- .50003	- .38956	- .31638	- .26419	- .22464	- .19332	- .16756	- .14585	- .12718
2.4	-1.00000	- .67561	- .50236	- .39272	- .32020	- .26849	- .22929	- .19824	- .17271	- .15117	- .13262
2.5	-1.00000	- .67679	- .50453	- .39566	- .32374	- .27248	- .23363	-.20286	- .17753	- .15616	- .13774
2.6	-1.00000	- .67788	- .50653	- .39840	- .32704	- .27621	- .23770	- .20719	- .18207	- .16084	- .14256
2.7	-1.00000	- .67889	- .50839	- .40095	- .33012	- .27970	- .24151	- .21125	- .18632	- .16527	- .14710
2.8	-1.00000	- .67984	- .51014	- .40334	- .33301	- .28298	- .24509	- .21507	- .19035	- .16944	- .15140
2.9	-1.00000	- .68073	- .51177	- .40558	- .33572	- .28606	- .24847	- .21868	- .19414	- .17338	- .15548
3.0	-1.00000	- .68156	- .51331	- .40769	- .33828	- .28897	- .25166	- .22209	- .19774	- .17713	- .15934
∞	-1.00000	- .71063	- .56794	- .48435	- .43383	- .40090	- .37800	- .36130	- .34855	- .33852	- .33043

Table 3. (continued) $c\,\Delta X(c,c') = \frac{c}{\pi}\int_0^1 \frac{T_{c'}-T_c}{1-k^2s^2}\,ds$

c' \ c	1.0	1.1	1.2	1.3	1.4	1.5	1.6	1.7	1.8	1.9	2.0
0	.28954	.28942	.28912	.28870	.28823	.28769	.28713	.28659	.28601	.28546	.28492
0.1	.26236	.26368	.26461	.26531	.26583	.26619	.26644	.26663	.26674	.26681	.26685
0.2	.23002	.23338	.23608	.23827	.24012	.24167	.24297	.24410	.24506	.24592	.24667
0.3	.19298	.19898	.20389	.20799	.21147	.21444	.21701	.21928	.22126	.22305	.22463
0.4	.15537	.16401	.17115	.17717	.18232	.18675	.19062	.19403	.19707	.19978	.20223
0.5	.12058	.13144	.14050	.14817	.15477	.16051	.16553	.17000	.17397	.17755	.18078
0.6	.08971	.10228	.11284	.12186	.12966	.13648	.14247	.14783	.15260	.15692	.16084
0.7	.06269	.07653	.08825	.09832	.10706	.11474	.12154	.12762	.13308	.13802	.14252
0.8	.03906	.05384	.06643	.07729	.08679	.09516	.10260	.10927	.11529	.12075	.12572
0.9	.01831	.03376	.04700	.05849	.06857	.07749	.08544	.09261	.09907	.10496	.11035
1.0	0.00000	.01594	.02965	.04160	.05214	.06148	.06985	.07741	.08425	.09050	.09622
1.1	- .01628	0.00000	.01407	.02637	.03725	.04694	.05564	.06351	.07066	.07720	.08320
1.2	- .03083	- .01432	0.00000	.01257	.02372	.03367	.04263	.05076	.05816	.06493	.07118
1.3	- .04393	- .02727	- .01276	0.00000	.01135	.02151	.03068	.03901	.04662	.05360	.06003
1.4	- .05577	- .03903	- .02440	- .01150	0.00000	.01032	.01966	.02816	.03593	.04307	.04966
1.5	- .06654	- .04975	- .03505	- .02205	- .01045	0.00000	.00946	.01809	.02600	.03328	.04001
1.6	- .07638	- .05936	- .04485	- .03179	- .02010	- .00957	0.00000	.00873	.01675	.02413	.03097
1.7	- .08541	- .06841	- .05389	- .04080	- .02905	- .01846	- .00881	0.00000	.00810	.01558	.02250
1.8	- .09373	- .07677	- .06226	- .04914	- .03738	- .02674	- .01704	- .00817	0.00000	.00755	.01456
1.9	- .10142	- .08452	- .07004	- .05693	- .04515	- .03448	- .02475	- .01582	- .00761	0.00000	.00706
2.0	- .10856	- .09172	- .07729	- .06419	- .05242	- .04173	- .03198	- .02301	- .01476	- .00711	0.00000
2.1	- .11519	- .09845	- .08405	- .07099	- .05923	- .04825	- .03877	- .02980	- .02151	- .01383	- .00668
2.2	- .12139	- .10473	- .09040	- .07738	- .06563	- .05467	- .04519	- .03619	- .02789	- .02019	- .01300
2.3	- .12718	- .11062	- .09635	- .08338	- .07166	- .06071	- .05123	- .04224	- .03394	- .02621	- .01961
2.4	- .13262	- .11615	- .10195	- .08903	- .07736	- .06643	- .05697	- .04798	- .03966	- .03194	- .02471
2.5	- .13774	- .12136	- .10724	- .09438	- .08274	- .07184	- .06240	- .05341	- .04510	- .03737	- .03013
2.6	- .14256	- .12627	- .11224	- .09943	- .08784	- .07697	- .06755	- .05858	- .05028	- .04255	- .03532
2.7	- .14710	- .13093	- .11696	- .10422	- .09267	- .08184	- .07245	- .06350	- .05522	- .04748	- .04025
2.8	- .15140	- .13533	- .12144	- .10877	- .09728	- .08648	- .07711	- .06819	- .05992	- .05220	- .04497
2.9	- .15548	- .13950	- .12570	- .11309	- .10165	- .09090	- .08157	- .07266	- .06441	- .05671	- .04949
3.0	- .15934	- .14347	- .12975	- .11720	- .10582	- .09512	- .08582	- .07694	- .06871	- .06102	- .05381
∞	- .33043	- .32352	- .31811	- .31331	- .30916	- .29525	- .30235	- .29953	- .29701	- .29474	- .29267

Table 3. (continued) $c\,\Delta\,X(c,c') = \frac{c}{\pi}\int_0^1 \frac{\tau_{c'}-\tau_c}{1-k^2 s^2}\,ds$

c' \ c	2.0	2.1	2.2	2.3	2.4	2.5	2.6	2.7	2.8	2.9	3.0	∞
0	.28492	.28436	.28382	.28331	.28280	.28232	.28185	.28137	.28095	.28049	.28008	.25000
0.1	.26685	.26683	.26679	.26674	.26667	.25660	.26651	.26640	.26632	.26620	.26610	.25000
0.2	.24667	.24731	.24788	.24840	.24884	.24927	.24964	.24996	.25029	.25055	.25080	.25000
0.3	.22463	.22605	.22732	.22850	.22955	.23054	.23143	.23224	.23304	.23372	.23438	.25000
0.4	.20223	.20443	.20643	.20827	.20994	.21151	.21294	.21426	.21551	.21664	.21773	.25000
0.5	.18078	.18371	.18638	.18884	.19108	.19319	.19511	.19690	.19860	.20014	.20162	.25000
0.6	.16084	.16440	.16765	.17066	.17341	.17600	.17836	.18058	.18267	.18459	.18642	.25000
0.7	.14252	.14661	.15037	.15384	.15704	.16003	.16280	.16539	.16782	.17007	.17222	.25000
0.8	.12572	.13028	.13447	.13835	.14191	.14527	.14837	.15128	.15402	.15656	.15898	.25000
0.9	.11035	.11528	.11982	.12404	.12795	.13161	.13500	.13818	.14119	.14399	.14665	.25000
1.0	.09622	.10147	.10632	.11083	.11502	.11893	.12259	.12601	.12925	.13227	.13514	.25000
1.1	.08320	.08873	.09384	.09859	.10301	.10716	.11103	.11466	.11810	.12132	.12437	.25000
1.2	.07118	.07693	.08225	.08723	.09185	.09619	.10025	.10406	.10767	.11105	.11427	.25000
1.3	.06003	.06597	.07149	.07663	.08143	.08594	.09016	.09413	.09790	.10143	.10480	.25000
1.4	.04966	.05577	.06144	.06673	.07168	.07634	.08070	.08480	.08871	.09237	.09585	.25000
1.5	.04001	.04625	.05204	.05748	.06255	.06733	.07182	.07604	.08006	.08384	.08744	.25000
1.6	.03097	.03733	.04324	.04878	.05396	.05885	.06344	.06777	.07188	.07576	.07946	.25000
1.7	.02250	.02895	.03496	.04060	.04587	.05085	.05554	.05995	.06416	.06813	.07191	.25000
1.8	.01456	.02108	.02716	.03287	.03823	.04329	.04805	.05256	.05683	.06088	.06475	.25000
1.9	.00706	.01365	.01981	.02559	.03101	.03613	.04097	.04554	.04988	.05399	.05793	.25000
2.0	0.00000	.00664	.01285	.01868	.02417	.02934	.03424	.03887	.04328	.04745	.05143	.25000
2.1	- .00668	0.00000	.00626	.01214	.01767	.02290	.02785	.03253	.03697	.04121	.04524	.25000
2.2	- .01300	- .00629	0.00000	.00592	.01149	.01676	.02175	.02648	.03097	.03525	.03933	.25000
2.3	- .01961	- .01227	- .00594	0.00000	.00561	.01091	.01594	.02071	.02523	.02955	.03367	.25000
2.4	- .02471	- .01795	- .01161	- .00563	0.00000	.00534	.01039	.01519	.01975	.02411	.02826	.25000
2.5	- .03013	- .02338	- .01701	- .01101	- .00536	0.00000	.00508	.00991	.01450	.01888	.02307	.25000
2.6	- .03532	- .02854	- .02216	- .01616	- .01048	- .00510	0.00000	.00485	.00947	.01388	.01808	.25000
2.7	- .04025	- .03348	- .02710	- .02107	- .01538	- .00999	- .00487	0.00000	.00464	.00907	.01330	.25000
2.8	- .04497	- .03819	- .03181	- .02578	- .02007	- .01467	- .00954	- .00465	0.00000	.00445	.00870	.25000
2.9	- .04949	- .04270	- .03632	- .03029	- .02458	- .01917	- .01404	- .00914	- .00446	0.00000	.00426	.25000
3.0	- .05381	- .04704	- .04066	- .03463	- .02891	- .02350	- .01834	- .01344	- .00875	- .00428	0.00000	.25000
∞	- .29267	- .29080	- .28910	- .28752	- .28607	- .28474	- .28350	- .28236	- .28127	- .26027	- .27933	0.00000

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